The Lagrange Inversion Formula (cont)

Following Wilf we consider the following functional equation

$$z = x\phi(z) \tag{1}$$

Can we solve for z as an explicit function of x? Can we find a closed formula for the sequence of coefficients, $[x^n]z(x)$? *Note:* The functional equation (1) implies $z(0) = 0$.

Theorem 1 (The Lagrange inversion formula (LIF)). Suppose that $W(z)$ and $\phi(z)$ are formal power series in z with $\phi(0) \neq 0$. Then there is a unique formal power series $z = z(x) = \sum_n z_n x^n$, satisfying (1). In addition, the value of $W(z(x))$ when expanded in a power series in x about $x = 0$ satisfies

$$n[x^n]W(z(x)) = [z^{n-1}]\{W'(z)\phi(z)^n\} \tag{2}$$

The simplest version of the theorem occurs when we take $W(z) = z$. In that case, (2) reduces to

$$n[x^n]z(x) = [z^{n-1}]\phi(z)^n \tag{3}$$

There are numerous proofs in the literature. We will present three proofs.

Note: If $z(x)$ is a formal power series about $x = 0$, we follow the standard convention that $[x^m]z(x) = 0$ whenever $m < 0$.

Proof (First Proof of LIF): We proceed by induction on $n \geq 0$. For the base case, both sides of (2) are clearly 0 whenever $n = 0$. Now suppose that (2) is true for $0 < m < n$. As we attempted to illustrate when verifying binomial inversion using LIF, it is enough to show that the following holds for all k.

$$n[x^n]z(x)^k = k[z^{n-1}]\{z^{k-1}\phi(z)^n\} \tag{4}$$

We consider a few special cases:

(i) $k = 0$: The right-hand side of (4) is clearly 0, and $n > 0$ implies that $n[x^n]z(x)^0 = n[x^n]\,1 = 0$.

(ii) $k > n$: Then $n - k < 0$ so that

$$n[x^n]z(x)^k \stackrel{(1)}{=} n[x^n]x^k\phi(z(x))^k = n[x^{n-k}]\phi(z(x))^k = 0$$

as we remarked above. The right-hand side is 0 for the same reason.

(iii) $k = n$: We have

$$\begin{aligned} n[x^n]z(x)^n &\stackrel{(1)}{=} n[x^n]x^n\phi(z(x))^n = n[x^0]\phi(z(x))^n = n\phi(z(0))^n \\ &= n\phi(0)^n \\ &= n[z^0]\phi(z)^n = n[z^{n-1}]\{z^{n-1}\phi(z)^n\} \end{aligned}$$

Now suppose that $0 < k < n$. Then

$$\begin{aligned}
n[x^n]z(x)^k &\overset{(1)}{=} n[x^{n-k}]\phi(z(x))^k \\
&\overset{(*)}{=} \frac{n}{n-k}[z^{n-k-1}]\left(\phi(z)^k\right)'\phi(z)^{n-k} \\
&= \frac{n}{n-k}[z^{n-k-1}]k\phi(z)^{k-1}\phi'(z)\phi(z)^{n-k} \\
&= \frac{k}{n-k}[z^{n-k}]nz\phi(z)^{n-1}\phi'(z) \\
&= \frac{k}{n-k}[z^{n-k}]zD_z\left(\phi(z)^n\right) \\
&\overset{(**)}{=} \frac{k}{n-k}(n-k)[z^{n-k}]\phi(z)^n \\
&= [z^{n-1}]kz^{k-1}\phi(z)^n
\end{aligned}$$

as desired. Notice that we were able to use the induction hypothesis (4) at step (*) since $n - k < n$ and that we invoked Wilf Rule 2 at step (**). □

Remark. The induction proof is essentially the one presented in the 2023 paper by Surya and Warnke, *Lagrange Inversion Formula by Induction.*

We will explore the other proofs in later lectures.

The Lagrange Inversion Formula - Applications

Example 2. In his 2009 paper, B. Sagan answered a conjecture by J. Propp. Suppose the sequence $\{a_n\}_{n\geq 0}$ counts the number of a certain type of colored triangulations of an n-gon. Then

$$a_N = \begin{cases} \dfrac{2^n}{2n+1}\dbinom{3n}{n} & \text{if } N = 2n, \\[2ex] \dfrac{2^{n+1}}{2n+2}\dbinom{3n+1}{n} & \text{if } N = 2n+1. \end{cases} \tag{5}$$

Sagan's proof proceeded as follows

$$a_{2n+1} = \sum_{j=0}^{n} a_{2j}a_{2n-2j} \tag{6}$$

$$a_{2n} = \sum_{j=0}^{2n-1} a_j a_{2n-1-j} \tag{7}$$

Now we consider the ordinary generating functions

$$E(x) = \sum_{n\geq 1} a_{2n}\, x^n \quad \text{and} \quad O(x) = \sum_{n\geq 0} a_{2n+1}\, x^n$$

First observe that, for example,

$$\begin{aligned} a_{2(3)} &= \sum_{j=0}^{2(3)-1} a_j a_{2(3)-1-j} \\ &= a_0a_5 + a_1a_4 + \cdots + a_5a_0 \\ &= 2\sum_{j=0}^{3-1} a_j a_{2(3)-1-j} \end{aligned}$$

So, in general we have

$$a_{2n} = 2\sum_{j=0}^{n-1} a_j a_{2n-1-j}$$

We leave it as an exercise to show that

$$2\sum_{j=0}^{n-1} a_j a_{2n-1-j} = 2\sum_{j=0}^{n-1} a_{2j+1}a_{2n-2j-2} = a_{2n} \tag{8}$$

Now

$$\begin{aligned} E(x) = \sum_{n\geq 1} a_{2n}\, x^n &\overset{(8)}{=} 2\sum_{n\geq 1}\sum_{j=0}^{n-1} a_{2j+1}a_{2n-2j-2}\, x^n \\ &= 2x\sum_{n\geq 1}\sum_{j=0}^{n-1} a_{2j+1}a_{2n-2j-2}\, x^{n-1} \\ &= 2x\sum_{n\geq 0}\sum_{j=0}^{n} a_{2j+1}a_{2n-2j}\, x^n \\ &= 2x\sum_{n\geq 0} a_{2n+1}\, x^n \sum_{n\geq 0} a_{2n}\, x^n \\ &= 2xO(x)(1+E(x)) \end{aligned}$$

That is,

$$E(x) = 2x(1+E(x))O(x) \tag{9}$$

We leave it as an exercise to show

$$O(x) = (1+E(x))^2 \tag{10}$$

Plugging (10) into (9) yields

$$E(x) = 2x(1 + E(x))^3 \tag{11}$$

This looks familiar (see Example 2 from Inversion Theorems - Part 2). So let $\phi(z) = 2(1 + z)^3$. Then $E(x) = x\phi(E(x))$ and we may invoke LIF. So by (3) we have

$$\begin{aligned}[x^n]E(x) &= \frac{1}{n}[z^{n-1}]\phi^n(z) \\ &= \frac{2^n}{n}[z^{n-1}](1+z)^{3n} \\ &= \frac{2^n}{n}\binom{3n}{n-1}\end{aligned}$$

which is equivalent to (5) when $N = 2n$. We leave the case $N = 2n + 1$ as an exercise.

The Lagrange Inversion Formula is especially adept when dealing with counting problems that arise when dealing with trees.

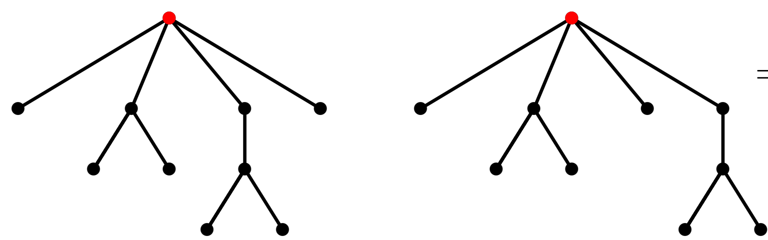
$\tau_1 =$  $= \tau_2$

Figure 1: Two distinct plane trees of size 10

A *tree* is an acyclic, connected graph. A tree is called *rooted* if one of its vertices is identified (the root). A *plane tree* (or ordered tree) is a rooted tree with a specified order assigned to the children of each vertex. Figure 1 shows two distinct plane trees τ_1 and τ_2 of order 10. The root is shown in red and there is an implied order for the four vertices that are adjacent to the root (its children). This implied order is left to right. And the order pattern continues with each descendant. We should also point out the size of a tree in these definitions is the number of nodes (vertices). We explore another way to define size in the exercises.

Now let $\mathcal{G}$ be the class of all plane trees where we once again measure size of each tree by the number of nodes it possesses and let $G(x)$ be the corresponding ogf. Also, let $\mathcal{Z} = \{\bullet\}$. It turns out that $\mathcal{G}$ satisfies the recursion

$$\mathcal{G} = \mathcal{Z} \times \mathrm{SEQ}(\mathcal{G}) \tag{12}$$

This translates to the following functional equation for its generating function.

$$G(x) = \frac{x}{1 - G(x)} \tag{13}$$

One than then show that

$$G(x) = \frac{1-\sqrt{1-4x}}{2} = x\,\frac{1-\sqrt{1-4x}}{2x} \tag{14}$$

$$= xC(x) \tag{15}$$

Here $C(x)$ is the ogf for the Catalan numbers. Now

$$[x^n]G(x) = [x^{n-1}]C(x) = c_{n-1} \tag{16}$$

In other words, $g_n = [x^n]G(x) = c_{n-1}$ are the shifted Catalan numbers. Now let $\phi(z) = (1-z)^{-1}$. Then (13) can be rewritten as

$$G(x) = x\phi(G(x))$$

and we may now apply the Lagrange Inversion formula to obtain

$$\begin{aligned}
[x^n]G(x) &= \frac{1}{n}[z^{n-1}](1-z)^{-n} \\
&= \frac{1}{n}[z^{n-1}]\sum_{k\geq 0}\binom{-n}{k}(-1)^k\,x^k \\
&= \frac{1}{n}[z^{n-1}]\sum_{k\geq 0}\binom{n-k-1}{k}\,x^k \\
&= \frac{1}{n}\binom{2n-2}{n-1} = c_{n-1}
\end{aligned}$$

as we saw above. However, one might argue that using LIF is often easier, even when an explicit form of the generating function is known (as in the case above), and the general recursive nature of many types of trees is especially amenable to this technique. We need a few definitions.

Definition 3. A k-ary tree is a plane tree (hence rooted) such the number of children of any node (vertex) is either 0 or k. Will use the notation $\mathcal{T}^{\{0,k\}}$ for the class of k-ary trees. *Note:* The classes $\mathcal{T}^{\{0,1\}}$, $\mathcal{T}^{\{0,2\}}$, and $\mathcal{T}^{\{0,3\}}$ are called unary, binary, and ternary trees, respectively.

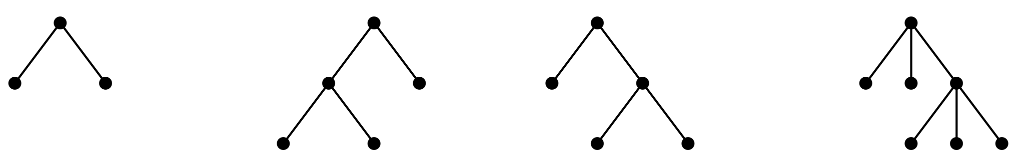


Figure 2: Three binary trees and one ternary tree

We display a few examples in Figure 2. Notice that we continue with convention of displaying the root at the top of the tree. Also, the size of a k-ary tree is the number of nodes. So the respective sizes of the 4 trees shown in Figure 2 are 3, 5, 5, and 7.

We can extend the above definition to include the following.

Definition 4. Let Ω be a finite subset of the natural numbers that contains 0. Then $\mathcal{T}^\Omega$ will be the class of plane trees such that the number of children at each node lies in Ω.

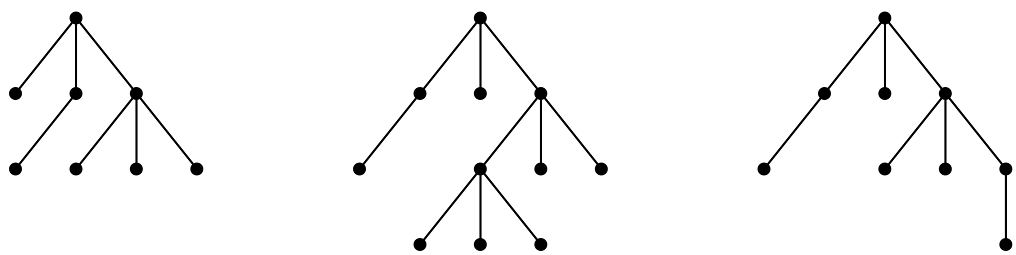


Figure 3: Three trees from the class $\mathcal{T}^{\{0,1,3\}}$

We will refer to such trees as Ω-restricted. Following Flajolet and Sedgewick, we define the characteristic function that encapsulates Ω by

$$\phi(z) = \sum_{k \in \Omega} z^k \tag{17}$$

For example, if $\Omega = \{0, 3\}$ then $\phi(z) = 1 + z^3$ is the characteristic function for ternary trees, and the characteristic function of $\Omega = \mathbb{N}$ is $\phi(z) = (1-z)^{-1}$, i.e., the characteristic function of unrestricted plane trees. We have the following proposition.

Proposition 5. The ordinary generating function $T^\Omega(x)$ of the class $\mathcal{T}^\Omega$ of Ω-restricted trees satisfies the following recursion

$$T^\Omega(x) = x\phi(T^\Omega(x)) \tag{18}$$

where ϕ is the characteristic of the set Ω as defined in (17). We also have

$$[x^n]T^\Omega(x) = \frac{1}{n}[z^{n-1}]\phi(z)^n \tag{19}$$

Proof: Notice that by the Lagrange Inversion formula, (19) follows immediately from (18), so it is enough to verify (18). If $\mathcal{A}$ is an Ω-restricted sequence, say $\mathcal{A} = \mathrm{SEQ}_\Omega(\mathcal{B})$ for some class $\mathcal{B}$, then its generating function is

$$A(x) = \phi(B(x))$$

Thus

$$\mathcal{T}^\Omega = \mathcal{Z} \times \mathrm{SEQ}_\Omega(\mathcal{T}^\Omega) \quad \Longrightarrow \quad T^\Omega(x) = x\phi(T^\Omega(x))$$

$\square$

Example 6. Find the counting sequence for the Ω-restricted class $\mathcal{T}^{\Omega}$ if $\Omega = \{0, 1, 3\}$. Let $T(x) = T^{\Omega}(x)$.

Now the characteristic function is $\phi(z) = 1 + z + z^3$, so by Proposition 5 (or the Lagrange Inversion formula),

$$\begin{aligned}
[x^n]T(x) &= \frac{1}{n}[z^{n-1}](1 + z + z^3)^n \\
&= \frac{1}{n}[z^{n-1}]\sum_{k=0}^{n}\binom{n}{k}z^{n-k}(1+z^2)^{n-k} \\
&= \frac{1}{n}[z^{n-1}]\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}z^{n-k+2j} \\
&= \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n-k}{\frac{k-1}{2}}
\end{aligned}$$

The first few terms in this sequence are $0, 1, 1, 1, 2, 5, 11, 24, 57, 141, 349, 871, \ldots$

Exercises

1. Let $g(x)$ be a formal power series and let p be a positive integer. Show that

$$[x^p]xD_x(g(x)) = p[x^p]g(x)$$

   Note: See the comments about Wilf Rule 2 at the end of the proof of Theorem 1.

2. Verify identity (8).

3. Let $E(x)$ and $O(x)$ be as defined in Example 2. Show that $O(x) = (1 + E(x))^2$. Also, prove that

$$[x^n]O(x) = \frac{2^{n+1}}{2n+2}\binom{3n+1}{n}$$

4. Sketch the 11 trees of size 6 from Example 6.

5. Sketch all trees of size 5 for the Ω-restricted trees $\mathcal{T}^\Omega$ listed below. Also, find the characteristic function $\phi(z)$, and use Proposition 5 (or the Lagrange Inversion formula) to find a closed form of the counting sequence of each class.

   (a) $\mathcal{T}^{\{0,2\}}$

   (b) $\mathcal{T}^{\{0,1,2,3\}}$