

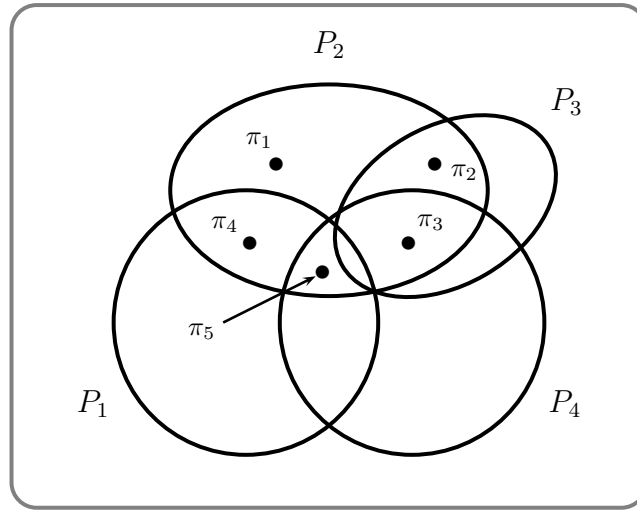


Figure 2: The Sieve Method

The Sieve Method

Let Γ be a set of objects and P be a set of properties that one or more of the objects possess. Let e_k denote the number of objects that have exactly k properties. How might we go about investigating this quantity?

Often it is easier to find out how many objects possess at least k properties. So let $S \subseteq P$ and let $N(\supseteq S)$ count the number of objects that have at least the properties in S . Now for a fixed $k \geq 0$ let

$$N_k = \sum_{|S|=k} N(\supseteq S) \quad (5)$$

Now let $\pi \in \Gamma$ and let $P(\pi)$ denote the subset of all of the properties that π possesses.

For example, in Figure 2 consider all of the subsets S of the properties $P = \{P_1, P_2, P_3, P_4\}$ with $|S| = 2$. Now $P(\pi_3) = \{P_2, P_3, P_4\}$ so that π_3 would contribute 3 times to N_2 . Likewise, π_5 contributes 3 times to N_2 . On the other hand, π_1 makes no contribution to N_k for $k > 1$ since $P(\pi_1) = \{P_2\}$.

Now let $j \geq k$ and suppose that $|P(\pi)| = j$. How much does π contribute to N_k ? It should be $\binom{j}{k}$. Thus

$$N_k = \sum_{j \geq k} \binom{j}{k} e_j$$

Now let $N(x)$ and $E(x)$ be the generating functions for N_k and e_j , respectively. Then

$$\begin{aligned} N(x) &= \sum_k N_k x^k = \sum_k \sum_{j \geq k} \binom{j}{k} e_j x^k \\ &= \sum_j e_j \left\{ \sum_k \binom{j}{k} x^k \right\} = \sum_j e_j (1+x)^j = E(x+1) \end{aligned}$$

Since we are really interested in the e_j 's, we usually write

$$E(x) = N(x-1) \tag{6}$$

Notice that the number of objects that have none of the given properties is

$$e_0 = E(0) = N(-1)$$