## 7.4 Partial Fractions

Consider the following integral.

(1)
$$\int \frac{13-2x}{x^2-x-2}\,dx$$

How might we evaluate this? Suppose that, by some good luck, we knew that

(2)
$$\frac{13-2x}{x^2-x-2} = \frac{3}{x-2} - \frac{5}{x+1}$$

We could then evaluate (1) as follows

$$\begin{aligned}\int \frac{13-2x}{x^2-x-2}\,dx &= 3\int \frac{1}{x-2}\,dx - 5\int \frac{1}{x+1}\,dx \\ &= 3\ln|x-2| - 5\ln|x+1| + C\end{aligned}$$

The right-hand side of (2) is called the “partial fraction” decomposition of

$$\frac{13-2x}{x^2-x-2}$$

# General Method

In this section we will learn to decompose rational expressions into simpler **partial fractions**, as we saw in (2). We will also discuss a few shortcuts.

To decompose the rational function $f(x)/d(x)$ as a sum of partial fractions, we need two things.

1. $f(x)/d(x)$ must be a proper fraction, i.e., $\deg(f) < \deg(d)$. If not, eliminate any common factors and use long division to rewrite the fraction as a quotient polynomial plus the remainder polynomial over the (possibly) new divisor polynomial.

   (3) $$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d_1(x)}$$

2. We must be able to factor the (new) divisor.

Now suppose that $f(x)/d(x)$ is a proper fraction and the factors of $d(x)$ are known. We proceed as follows.

1. Suppose that $x-a$ is a factor of $d(x)$ and $(x-a)^m$ is the highest power of $x-a$ that divides $d(x)$. Then we assign the following to the sum of partial fractions
$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_m}{(x-a)^m}$$

   We must repeat the above procedure for each of the linear factors of $d(x)$.

2. Now suppose that $x^2+bx+c$ is an (irreducible) quadratic factor of $d(x)$ and that $(x^2+bx+c)^n$ is the highest power of $x^2+bx+c$ that divides $d(x)$. Then, as above, we assign the following to the sum of partial fractions.
$$\frac{B_1x+C_1}{x^2+bx+c}+\frac{B_2x+C_2}{(x^2+bx+c)^2}+\cdots+\frac{B_nx+C_n}{(x^2+bx+c)^n}$$

   Once again, we repeat the above procedure for each of the quadratic factors of $d(x)$.

3. Now we sum all of the partial fractions from steps 1 and 2 and equate them to $f(x)/d(x)$. Now we clear the denominators and solve the resulting (linear) system of equations by collecting like powered terms and equating them to the corresponding powers in the polynomial $f(x)$.

**Example 1. Partial Fraction Decomposition**

Find a partial fraction decomposition of each of the following.

a. $\dfrac{13-2x}{x^2-x-2}$

So according to step 1 above, we must rewrite this as

$$\frac{13-2x}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

Now we clear the denominators, equate like coefficients, and solve for $A$ and $B$. Thus

$$13-2x = A(x+1) + B(x-2)$$
$$\Longrightarrow 13 = A - 2B$$
$$-2x = (A+B)x \quad \text{or} \quad -2 = A + B$$

It is easy to solve this 2 by 2 system to conclude that
$A = 3$ and $B = -5$, as we saw above.

b. $\dfrac{5x^2-3x-2}{(x-5)(x+1)^2}$

The candidate decomposition is

$$\frac{5x^2-3x-2}{(x-5)(x+1)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-5}$$

Clearing denominators yields

$$\begin{aligned}5x^2-3x-2&=A(x+1)(x-5)+B(x-5)+C(x+1)^2\\&=Ax^2-4Ax-5A+Bx-5B+Cx^2+2Cx+C\end{aligned}$$

Now equating like terms leads to the 3 by 3 system

$$\begin{aligned}-2&=-5A-5B+C\\-3&=-4A+B+2C\\5&=A+C\end{aligned}$$

The solution of this system is $A=2,\ B=-1,$ and $C=3$. In other words,

$$\frac{5x^2-3x-2}{(x-5)(x+1)^2}=\frac{2}{x+1}+\frac{-1}{(x+1)^2}+\frac{3}{x-5}$$

Now we can easily evaluate the following integral.

$$\begin{aligned}\int \frac{5x^2-3x-2}{(x-5)(x+1)^2}\,dx &= 2\int \frac{1}{x+1}\,dx \\ &\quad - \int \frac{1}{(x+1)^2}\,dx + 3\int \frac{1}{x-5}\,dx \\ &= 2\ln|x+1| + \frac{1}{x+1} + 3\ln|x-5| + C\end{aligned}$$

c. $\dfrac{x^2-4x-12}{(x+5)\left(x^2+3x+1\right)}$

$$\frac{x^2-4x-12}{(x+5)\left(x^2+3x+1\right)}=\frac{A}{x+5}+\frac{Bx+C}{x^2+3x+1}$$

Thus

$$\begin{aligned}x^2-4x-12&=A\left(x^2+3x+1\right)+(x+5)(Bx+C)\\&=Ax^2+3Ax+A+Bx^2+5Bx+Cx+5C\end{aligned}$$

Equating like terms leads to the 3 by 3 system

$$\begin{aligned}-4&=3A+5B+C\\-12&=A+5C\\1&=A+B\end{aligned}$$

which has the solution $A=3$, $B=-2$, and $C=-3$. Or

$$\frac{x^2-4x-12}{(x+5)\left(x^2+3x+1\right)}=\frac{3}{x+5}-\frac{2x+3}{x^2+3x+1}$$

We now leave it as an easy exercise to evaluate

$$\int\frac{x^2-4x-12}{(x+5)\left(x^2+3x+1\right)}\,dx$$

# Other Methods

As we saw earlier, partial fraction decomposition can be rather tedious. Consider the following example.

**Example 2. Assigning Numerical Values**

Find the partial fraction decomposition of

$$\frac{2x^4 - 11x^3 - 13x^2 - 97x + 71}{(x-5)(x-1)(x+3)\left(x^2+2\right)}$$

Let $f(x) = 2x^4 - 11x^3 - 13x^2 - 97x + 71$. Now follow steps 1 and 2 on page 3 to obtain.

$$\frac{2x^4 - 11x^3 - 13x^2 - 97x + 71}{(x-5)(x-1)(x+3)\left(x^2+2\right)}$$

$$= \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{x-5} + \frac{Dx+E}{x^2+2}$$

Clearing fractions yields

$$\begin{aligned} f(x) = 2x^4 - 11x^3 - 13x^2 - 97x + 71 \\ &= A(x+3)(x-5)(x^2+2) \\ &+ B(x-1)(x-5)(x^2+2) \\ &+ C(x-1)(x+3)(x^2+2) \\ &+ (Dx+E)(x-1)(x+3)(x-5) \end{aligned}$$

Now substitute several convenient values for $x$ and solve the resulting equations. For example, if $x = 1$ we have

$$\begin{aligned} -48 &= f(1) \\ &= A(4)(-4)(3) + B \cdot 0 + C \cdot 0 + (Dx + E) \cdot 0 \\ \implies A &= 1 \end{aligned}$$

If $x = -3$ we have

$$\begin{aligned} 704 &= f(-3) \\ &= B(-4)(-8)(11) \\ \implies B &= 2 \end{aligned}$$

If $x = 5$ then

$$\begin{aligned} -864 &= f(5) \\ &= C(4)(8)(27) \\ \Longrightarrow C &= -1 \end{aligned}$$

Now $A, B$, and $C$ are known. If $x = 0$ then

$$\begin{aligned} 71 &= f(0) \\ &= 1(-30) + 2(10) - 1(-6) + E(15) \\ \Longrightarrow E &= 5 \end{aligned}$$

Finally, let $x = -1$. Then

$$\begin{aligned} 168 &= f(-1) \\ &= 1(-36) + 2(36) - 1(-12) + (5 - D)(24) \\ \Longrightarrow D &= 0 \end{aligned}$$

It follows that

$$\frac{2x^4 - 11x^3 - 13x^2 - 97x + 71}{(x-5)(x-1)(x+3)\left(x^2+2\right)} = \frac{1}{x-1} + \frac{2}{x+3} + \frac{-1}{x-5} + \frac{5}{x^2+2}$$

It turns out that the method employed above can be further simplified if the divisor, $d(x)$, has only linear factors.

**Example 3. Heaviside "Cover-up" Method**

Rewrite the following as a sum of partial fractions.

$$\frac{7x^3+12x^2-53x-14}{(x-5)(x-1)(x+2)(x+3)}$$

We proceed as usual, but we don't clear the denominators.

$$\frac{7x^3+12x^2-53x-14}{(x-5)(x-1)(x+2)(x+3)} = \frac{A}{x-5}+\frac{B}{x-1}+\frac{C}{x+2}+\frac{D}{x+3}$$

Now if we multiply the above equation by $x-5$ we get

$$\frac{7x^3+12x^2-53x-14}{(x-1)(x+2)(x+3)} = A+\frac{B(x-5)}{x-1}+\frac{C(x-5)}{x+2}+\frac{D(x-5)}{x+3}$$

Now substitute $x=5$ into the result to get

$$\frac{7(5)^3+12(5)^2-53(5)-14}{(5-1)(5+2)(5+3)} = A+0+0+0$$

$$\Longrightarrow A=4$$

We can accomplish the same thing by “covering-up” the factor $x-5$ on the left-hand side and ignoring the second, third and fourth terms on the right-hand side. Let’s use this technique to find $B, C,$ and $D$.

To find $B$ we must “cover” the $x-1$ factor on the left-hand side. Thus

$$\frac{7x^3+12x^2-53x-14}{(x-5)\underbrace{(x-1)}_{\text{Cover}}(x+2)(x+3)}$$

$$=\frac{A}{\underbrace{x-5}_{\text{Cover}}}+\frac{B}{\underbrace{x-1}_{\text{Cover}}}+\frac{C}{\underbrace{x+2}_{\text{Cover}}}+\frac{D}{\underbrace{x+3}_{\text{Cover}}}$$

Now let $x=1$ to find $B$.

$$B=\frac{7(1)^3+12(1)^2-53(1)-14}{((1)-5)\underbrace{(x-1)}_{\text{Cover}}((1)+2)((1)+3)}$$

$$=1$$

To find $C$ we let $x=-2$ and cover the $x+2$ factor.

$$\frac{7(-2)^3+12(-2)^2-53(-2)-14}{((-2)-5)((-2)-1)\underbrace{(x+2)}_{\text{Cover}}((2)+3)}$$

$$=\frac{A}{\underbrace{x-5}_{\text{Cover}}}+\frac{B}{\underbrace{x-1}_{\text{Cover}}}+\frac{C}{\underbrace{x+2}_{\text{Cover}}}+\frac{D}{\underbrace{x+3}_{\text{Cover}}}$$

$$\Longrightarrow C=4$$

Finally, let $x = -3$ and cover the $x + 3$ factor.

$$\frac{7(-3)^3 + 12(-3)^2 - 53(-3) - 14}{((-3) - 5)((-3) - 1)((-3) + 2)\underbrace{(x + 3)}_{\text{Cover}}}$$

$$= \frac{A}{\underbrace{x - 5}_{\text{Cover}}} + \frac{B}{\underbrace{x - 1}_{\text{Cover}}} + \frac{C}{\underbrace{x + 2}_{\text{Cover}}} + \frac{D}{\underbrace{x + 3}_{\text{Cover}}}$$

$$\implies D = -2$$

It follows that

$$\frac{7x^3 + 12x^2 - 53x - 14}{(x - 5)(x - 1)(x + 2)(x + 3)}$$

$$= \frac{4}{x - 5} + \frac{1}{x - 1} + \frac{4}{x + 2} + \frac{-2}{x + 3}$$

**Example 4.** Evaluate

$$\int \frac{7x^3 + 12x^2 - 53x - 14}{(x-5)(x-1)(x+2)(x+3)}\,dx$$

$$\begin{aligned}
\int \frac{7x^3 + 12x^2 - 53x - 14}{(x-5)(x-1)(x+2)(x+3)}\,dx \\
&= \int \frac{4}{x-5}\,dx + \int \frac{1}{x-1}\,dx + \\
&\int \frac{4}{x+2}\,dx + \int \frac{-2}{x+3}\,dx \\
&= 4\ln|x-5| + \ln|x-1| + 4\ln|x+2| - 2\ln|x+3| + C
\end{aligned}$$