

Date	Section	Exercises** (QC - Quick Check and CE - Class Exercises)
11/17*	8.2	CE - 31, 32
11/19*	8.2	CE - 24, 45, 46
11/21*	Inversions	See below.
11/24*	8.2	CE - See below.
11/26*	Inversions	See below.
12/01*	Inversions	See below.
12/03*	Inversions	See below.
12/05*	Inversions	See below.

11/17

1. Extend Rule 3' to a product of 3 exponential generating functions. Verify your formula.

Solution:

Let $A \xrightarrow{\text{egf}} \{a_n\}_n$, $B \xrightarrow{\text{egf}} \{b_n\}_n$, and $C \xrightarrow{\text{egf}} \{c_n\}_n$. Then

$$\begin{aligned}
 A(x)B(x)C(x) &= \sum_{n \geq 0} \sum_k \underbrace{\binom{n}{k} a_k b_{n-k}}_{h_n} \frac{x^n}{n!} C(x) = \sum_{n \geq 0} h_n \frac{x^n}{n!} C(x) \\
 &= \sum_{n \geq 0} \sum_j \binom{n}{j} h_j c_{n-j} \frac{x^n}{n!} \\
 &= \sum_{n \geq 0} \sum_j \binom{n}{j} \sum_k \binom{j}{k} a_k b_{j-k} c_{n-j} \frac{x^n}{n!} \\
 &= \sum_{n \geq 0} \sum_{j=0}^n \sum_{k=0}^j \frac{n!}{k!(j-k)!(n-j)!} a_k b_{j-k} c_{n-j} \frac{x^n}{n!} \\
 &= \sum_{n \geq 0} \sum_{i+j+k=n} \frac{n!}{i!j!k!} a_i b_j c_k \frac{x^n}{n!}
 \end{aligned}$$

2. Recall that $\pi \in \mathfrak{S}_n$ is called an involution if $\pi^2 = \text{id}$. Let i_n count the number of involutions in \mathfrak{S}_n (the set of all permutations on $[n]$) and let $i_0 = 1$.

(a) Show that $i_1 = 1$ and for $n \geq 0$,

$$i_{n+2} = i_{n+1} + (n+1)i_n \quad (1)$$

Solution:

An involution must consist entirely of 1-cycles and 2-cycles. Now the left-hand side counts the number of involutions on $[n+2]$. For the right-hand side, there are i_{n+1} involutions with $n+2$ in a 1-cycle. Otherwise, there are $\binom{n+1}{1} = n+1$ ways to choose the element paired with $n+2$ and i_n ways to permute the remaining items (as an involution). So by the product rule, there are $(n+1)i_n$ ways that $n+2$ can be in a 2-cycle. Since these cases are mutually exclusive, the result now follows by the sum rule.

(b) Show that

$$\sum_{n \geq 0} i_n \frac{x^n}{n!} = e^{x+x^2/2}$$

Solution:

Let $A(x) = \sum_n i_n x^n/n!$. According to the Wilf rules, the recursion (1) is equivalent to the following differential equation

$$\begin{aligned} A''(x) &= A'(x) + (xD + I)A(x) \quad (D = \text{derivative operator and } I = \text{identity map}) \\ &= (x+1)A'(x) + A(x) = D((x+1)A(x)) \end{aligned}$$

Integrating both sides yields

$$A'(x) = (x+1)A(x) + C \quad (\text{but } C = 0 \text{ since } A'(0) = A(0) = 1)$$

Rearranging and integrating gives

$$\begin{aligned} \frac{A'(x)}{A(x)} &= 1 + x \\ \ln A(x) &= x + x^2/2 + C \quad (\text{and once again } C = 0 \text{ since } A(0) = 1) \end{aligned}$$

The result now follows.

3. Let $\{f_n\}_{n \geq 0}$ be a sequence and let Δ be the forward difference operator. That is, $\Delta f_j = f_{j+1} - f_j$. Show that

$$\Delta^n f_0 = \sum_k \binom{n}{k} f_k (-1)^{n-k} \quad (2)$$

Note: $\Delta^n f_k = \Delta \Delta^{n-1} f_k$ and $\Delta^0 f_k = f_k$.

Solution:

We induct on n . Clearly (2) holds when $n = 0$ since both sides produce f_0 . Now suppose that (2) holds. Then

$$\begin{aligned} \Delta^{n+1} f_0 &= \Delta \Delta^n f_0 = \Delta \sum_{k=0}^n \binom{n}{k} f_k (-1)^{n-k} \\ &= \sum_{k=0}^n \binom{n}{k} (f_{k+1} - f_k) (-1)^{n-k} \\ &= \sum_{k=0}^n \binom{n}{k} f_{k+1} (-1)^{n-k} - \sum_{k=0}^n \binom{n}{k} f_k (-1)^{n-k} \\ &= f_{n+1} + \sum_{k=1}^n \binom{n}{k-1} f_k (-1)^{n+1-k} + \sum_{k=1}^n \binom{n}{k} f_k (-1)^{n+1-k} + (-1)^{n+1} f_0 \\ &= f_{n+1} + \sum_{k=1}^n \left(\binom{n}{k-1} + \binom{n}{k} \right) f_k (-1)^{n+1-k} + (-1)^{n+1} f_0 \\ &= f_{n+1} + \sum_{k=1}^n \binom{n+1}{k} f_k (-1)^{n+1-k} + (-1)^{n+1} f_0 \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} f_k (-1)^{n+1-k} \end{aligned}$$

as expected.

11/19

1. Let $c_0 = 1$ and for $n > 0$ let c_n count the number of n -permutations in which each cycle is colored red, green, or blue.

(a) Find a sum formula for c_n .

Solution:

Let $\pi \in \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$. Then π can be colored in 3^k ways. So by the product rule, there are $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] 3^k$ ways to color n -permutations that consist of exactly k cycles. Summing across k yields

$$c_n = \sum_k \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] 3^k$$

(b) Find a simple factorial formula for c_n .

Solution:

Manual computation using the above formula produces the sequence 1, 3, 12, 60, ... So we guess $c_n = (n+2)!/2$, $n \geq 0$. Fortunately, we don't have to guess. In Math 481 we showed that

$$\sum_k \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] x^k = x^{\overline{n}} = x(x+1) \cdots (x+n-1)$$

After the substitution $x = 3$, we obtain

$$\begin{aligned} c_n &= 3(3+1) \cdots (3+n-1) \\ &= \frac{2}{2} \frac{3(3+1) \cdots (2+n)}{1} = \frac{(n+2)!}{2} \end{aligned}$$

(c) Let $C(x) = \sum_n c_n x^n / n!$. Find the closed form of $C(x)$.

Solution:

$$\begin{aligned} C(x) &= \sum_{n \geq 0} \frac{(n+2)!}{2} \frac{x^n}{n!} \\ &= \frac{1}{2} \sum_{n \geq 0} (n+2)(n+1)x^n \\ &= D^2 \left(\frac{1}{1-x} \right) = \frac{1}{(1-x)^3} \end{aligned}$$

(d) Now let $a_0 = a_1 = 1$ and let $a_{n+2} = c_n$ for $n \geq 0$. Find the closed form for $A(x) = \sum_n a_n x^n / n!$. *Note:* I will explain the reason for this part later.

Solution:

According to the Wilf rules, $A''(x) = C(x)$. It follows that $A(x) = (1-x)^{-1}$.

2. A coach wishes to break up her n -member team into 3 practice squads. Players on squad A will wear either red, white, or blue jerseys, those on squad B will wear yellow or green jerseys, and squad C players will wear black jerseys. Let $t_0 = 1$ and for $n > 0$, let t_n count the number of ways that she can do this.

- (a) Find a simple formula for t_n .

Solution:

There are 6 jersey colors, so this should just be 6^n .

- (b) Let $T(x) \xleftrightarrow{\text{egf}} \{t_n\}$. Find the closed form of $T(x)$ and use it to confirm your answer in part (a).

Solution:

Let i, j , and k be the number of players resp. on squad A, squad B, and squad C. Then

$$t_n = \sum_{i+j+k=n} \frac{n!}{i!j!k!} 3^i 2^j 1^k$$

So by the Wilf rules, we must have

$$\begin{aligned} T(x) &= \sum_n t_n \frac{x^n}{n!} = \sum_n 3^n \frac{x^n}{n!} \sum_n 2^n \frac{x^n}{n!} \sum_n \frac{x^n}{n!} \\ &= e^{3x} e^{2x} e^x = e^{6x} \end{aligned}$$

as expected.

- (c) In addition to the initial conditions, suppose also that squad B has a captain and players on squad C wear numbered black jerseys. Find the closed form for $T(x)$ in this case.

Solution:

$$t_n = \sum_{i+j+k=n} \frac{n!}{i!j!k!} 3^i j 2^j k!$$

So by the Wilf rules, we must have

$$\begin{aligned} T(x) &= \sum_n t_n \frac{x^n}{n!} = \sum_n 3^n \frac{x^n}{n!} \sum_n n 2^n \frac{x^n}{n!} \sum_n \frac{x^n}{n!} \\ &= e^{3x} 2x e^{2x} \frac{1}{1-x} = \frac{2x e^{5x}}{1-x} \end{aligned}$$

The first few terms of this [sequence](#) are

$$0, 2, 24, 222, 1888, 15690, 131640, 1140230, 10371840, \dots$$

11/21 Prove identities (3) and (4) below.

Hint: Earlier in the semester we showed $m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \sum_{k=0}^m \binom{m}{k} k^n (-1)^{m-k}$, provided $n \geq m \geq 0$.

$$\sum_n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!} \quad (3)$$

$$\sum_n \left[\begin{matrix} n \\ k \end{matrix} \right] \frac{x^n}{n!} = \frac{1}{k!} \left(\ln \frac{1}{1-x} \right)^k \quad (4)$$

(a) Let $S_k(x) = \sum_{n \geq 0} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{x^n}{n!}$.

Solution:

We proceed by induction on k . For $k = 0$, we have

$$\begin{aligned} S_0(x) &= \sum_{n \geq 0} \left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} \frac{x^n}{n!} \\ &= \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\} \frac{x^0}{0!} + \left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\} \frac{x^1}{1!} + \cdots + \left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} \frac{x^n}{n!} + \cdots \\ &= 1 + 0 + 0 + \cdots \end{aligned}$$

in agreement with (3) and the base case is established. Now suppose (3) holds for all $j < k$. Then

$$\begin{aligned} S'_k(x) &= \sum_{n \geq 0} \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} \frac{x^n}{n!} && \text{(Wilf Rule 1')} \\ &= k \sum_{n \geq 0} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{x^n}{n!} + \sum_{n \geq 0} \left\{ \begin{matrix} n \\ k-1 \end{matrix} \right\} \frac{x^n}{n!} && \text{(by recursion)} \\ &= k S_k(x) + \frac{(e^x - 1)^{k-1}}{(k-1)!} && \text{(by induction)} \end{aligned}$$

Rearranging produces the differential equation

$$S'_k(x) - k S_k(x) = \frac{(e^x - 1)^{k-1}}{(k-1)!}$$

which can be evaluated by elementary techniques. We try multiplying by the integrating factor e^{-kx} to obtain

$$D_x (e^{-kx} S_k(x)) = \frac{(1 - e^{-x})^{k-1}}{e^x (k-1)!}$$

Integrating both sides produces

$$e^{-kx} S_k(x) = \frac{(1 - e^{-x})^k}{k!} + C \quad (\text{but } C = 0 \text{ since } S_k(0) = 0)$$

Now this last equation is equivalent to (3).

Remark. Notice that $S_1(x) = \sum_{n \geq 1} \left\{ \begin{matrix} n \\ 1 \end{matrix} \right\} \frac{x^n}{n!} = \sum_{n \geq 1} \frac{x^n}{n!} = e^x - 1$. One could then determine that $S_2(x) = (e^x - 1)^2/2$, as we do in part (b) below, to “guess” the general formula in (3).

(b) Let $C_k(x) = \sum_{n \geq 0} \binom{n}{k} \frac{x^n}{n!}$.

Solution:

We outline the proof of (4) below and assume that we must try to guess the general formula as given in (4).

(i) First recall that $\binom{n}{1} = (n-1)!$. Thus

$$C_1(x) = \sum_n \binom{n}{1} \frac{x^n}{n!} = \sum_{n \geq 1} \frac{x^n}{n}$$

It follows that

$$C_1'(x) = \sum_{n \geq 1} x^{n-1} = \frac{1}{1-x}$$

so that

$$C_1(x) = \ln \frac{1}{1-x}$$

(ii) Before we try to guess a general pattern, let's try to find the closed form of $C_2(x)$. Taking derivatives in part (i) turned out to be useful. If we apply Wilf Rule 1' together with the recursion formula for $\binom{n}{k}$, we obtain

$$\begin{aligned} C_2'(x) &= \sum_n \binom{n+1}{2} \frac{x^n}{n!} \\ &= \sum_n n \binom{n}{2} \frac{x^n}{n!} + \sum_n \binom{n}{1} \frac{x^n}{n!} \\ &= xC_2'(x) + C_1(x) \end{aligned}$$

Rearranging yields the differential equation,

$$C_2'(x) = \frac{1}{1-x} \ln \frac{1}{1-x}$$

which admits the [solution](#),

$$C_2(x) = \frac{1}{2} \left(\ln \frac{1}{1-x} \right)^2$$

(iii) We claim that the general form appears to be

$$C_k(x) = \frac{1}{k!} \left(\ln \frac{1}{1-x} \right)^k$$

The proof of this fact is nearly identical to part (ii) and we leave it as an exercise. See also part (a) above.

11/24

1. Suppose the sequence $\{a_n\}_{n \geq 0}$ satisfies the following recursion. $a_0 = a_1 = 1$, $a_2 = 2$ and for $n > 2$

$$a_{n+1} = (n+1)a_n - \binom{n}{2}a_{n-2}$$

The first few terms of this sequence are 1, 1, 2, 5, 17, 73, ... Show the exponential generating function $A(x) = \sum_n a_n x^n/n!$ satisfies the ordinary differential equation

$$(1-x)A'(x) = \left(1 - \frac{x^2}{2}\right)A(x)$$

and is given by

$$A(x) = \frac{e^{x/2+x^2/4}}{\sqrt{1-x}}$$

2. Find the closed form of the exponential generating function $F_k(x) = \sum_{n \geq 0} \binom{n}{k} \frac{x^n}{n!}$.

11/26

1. Read the first two pages of Sagan's [paper](#).
 2. Show the following. *Hint:* For (5) use the generalized Binomial theorem.

$$\frac{1}{\sqrt{1-4x}} = \sum_{n \geq 0} \binom{2n}{n} x^n \quad (5)$$

$$\left(\frac{1 - \sqrt{1-4x}}{2x}\right)^k = \sum_{n \geq 0} \frac{k(2n+k-1)!}{n!(n+k)!} x^n \quad (6)$$

Solution:

For (5) we have

$$\frac{1}{\sqrt{1-4x}} = (1 + (-4x))^{-1/2} = \sum_{n \geq 0} \binom{-1/2}{n} (-4x)^n = \dots$$

We leave the details to the student.

For (6), we let $C(x) = (1 - \sqrt{1-4x})/(2x)$ and let $z(x) = C(x) - 1$. Then as we have shown before (see [Example 2](#)),

$$z = x(1+z)^2 = x\phi(z) \quad (7)$$

Now let $W(z) = (1+z)^k$, then by the Lagrange Inversion formula

$$\begin{aligned} [x^n]C(x)^k &= [x^n]W(z(x)) \\ &= \frac{1}{n} [z^{n-1}]W'(z)\phi(z)^n \\ &= \frac{k}{n} [z^{n-1}](1+z)^{k-1}(1+z)^{2n} \\ &= \frac{k}{n} [z^{n-1}](1+z)^{2n+k-1} \\ &= \frac{k}{n} \binom{2n+k-1}{n-1} \end{aligned}$$

12/01

1. Let $m_0 = 1$ and for $n > 0$, suppose that

$$m_n = m_{n-1} + \sum_{k=2}^n m_{k-2} m_{n-k} \quad (8)$$

Show that if $M(x) = \sum_{n \geq 0} m_n x^n$, then $M(x)$ satisfies the functional equation

$$M(x) - 1 = xM(x) + x^2M(x)^2 \quad (9)$$

Solution:

For $n \geq 2$, we have

$$[x^n](M(x) - 1) = m_n$$

and

$$\begin{aligned} [x^n](xM(x) + x^2M(x)^2) &= m_{n-1} + [x^{n-2}]M(x)^2 \\ &= m_{n-1} + [x^{n-2}] \sum_{p \geq 0} \sum_{k=0}^p m_k m_{p-k} x^p \\ &= m_{n-1} + \sum_{k=0}^{n-2} m_k m_{n-2-k} \\ &= m_{n-1} + \sum_{k=2}^n m_{k-2} m_{n-k} \end{aligned}$$

So by (8),

$$[x^n](M(x) - 1) = [x^n](xM(x) + x^2M(x)^2)$$

for $n \geq 2$. The cases when $n \in \{0, 1\}$ are trivial and are left as exercises. The result now follows.

2. Find the sum of the first n terms in the binomial expansion of

$$\left(1 - \frac{1}{2}\right)^{-n} = \sum_{k \geq 0} \binom{-n}{k} \left(\frac{-1}{2}\right)^k = \sum_{k \geq 0} \binom{n+k-1}{k} 2^{-k} \quad (10)$$

For example, when $n = 4$ the sum is $1 + 4/2 + 10/4 + 20/8 = 8$. *Hint:* Use LIF.

Solution:

Let d_n denote the left-hand side of (10). Then clearly, $d_n = 2^n$ and it is easy to guess that the sum of the first n terms of the right-hand sum is 2^{n-1} .

3. Let $J(x) = (1+x)^2/(2+x)$. Show that for all $n \in \mathbb{P}$

$$[x^{n-1}] \frac{J(x)^n}{1+x} = \frac{1}{2}$$

12/03

- Let $\{a_n\}_{n \geq 0} \subset \mathbb{R}$ with $a_0 \neq 0$. Find a sum formula for $[z^n] \left(\sum_{k=0}^N a_k z^k \right)^n$ for $N = 2$ and $N = 3$. Do you see a pattern?
- Let $\mathcal{T} = \mathcal{T}^\Omega$ where $\Omega = \{0, 1, 3\}$. However, this time we measure the size of each tree by the number of edges. Let $T(x)$ be the ordinary generating function for \mathcal{T} . Find a sum formula for $[x^n]T(x)$.
- Schröder numbers** - In 1870, the German mathematician Ernst Schröder asked the following question. In how many ways can n identical variables be “bracketed”? We give a recursive definition: x is a bracketing. And for $k \geq 2$, if $\delta_1, \delta_2, \dots, \delta_k$ are bracketed expressions, then so is $(\delta_1 \cdot \delta_2 \cdots \delta_k)$. For example, $x, (xx),$ and $(x(xx))$ are bracketed expressions and $(xx), (x(xx)), ((xx)x)$ are the three bracketings of size 3.

Now let $s_0 = 0$ and for $n > 0$, let s_n count the number of bracketings for n identical variables. If $S(x) \xrightarrow{\text{ogf}} \{s_n\}_{n \geq 0}$, show that

$$S(x) = x + \frac{S(x)^2}{1 - S(x)} \quad (11)$$

Although the Lagrange Inversion formula does not directly apply to (11), one can solve the equation to conclude

$$S(x) = \frac{1 + x - \sqrt{1 - 6x + x^2}}{4} \quad (12)$$

The counting sequence of $S(x)$ begins with 0, 1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859, ...

Answer the following questions about these Schröder numbers.

- List the 11 bracketings of size 4.
- Use (12) to show that

$$[x^n]S(x) = \frac{\delta_0(n) + \delta_1(n) + \sum_{k \geq 0} \binom{1/2}{k} \binom{k}{2k-n} (-6)^{2k-n}}{4}$$

- Rearrange the defining equation for $S(x)$ given in (11) so that the Lagrange Inversion formula can be applied. What is $\phi(z)$?
- Now use the Lagrange Inversion formula to show that

$$[x^n]S(x) = \frac{1}{n} \sum_{k \geq 0} \binom{2n - k - 2}{n - 1} \binom{n - 2}{k}$$

12/05

- Show that $\left[\begin{smallmatrix} n+1 \\ 2 \end{smallmatrix} \right] = n! \sum_{k=1}^n \frac{1}{k}$. *Hint:* See (4) from 11/21 above.
- Let $a_n = \sum_k (-1)^{n-k} \binom{n}{k} b_k$. Use a CAS and/or problem 1 from [Quiz 11](#) to find the first 6 terms in each of the following sequences. Also, use the [OEIS](#) to determine whether any of these new sequences are “interesting”.

(a) b_n are the Bell numbers. That is, $b_n = 1, 1, 2, 5, 14, 42, 132, 429, \dots$

(b) b_n are the Motzkin numbers, $b_n = 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, \dots$

Note: The ordinary generating function for the Motzkin numbers is $\frac{1 - \sqrt{(1-x)^2 - 4x^2}}{2x^2}$.

(c) b_n are the Schröder numbers, $b_n = 1, 1, 3, 11, 45, 197, 903, 4279, \dots$

Note: The ordinary generating function for the Schröder numbers is $\frac{1 + x - \sqrt{1 - 6x + x^2}}{4x}$.

(d) b_n are the Riordan numbers, $b_n = 1, 0, 1, 1, 3, 6, 15, 36, 91, 232, 603, \dots$

Note: The ordinary generating function for the Riordan numbers is $\frac{1 + x - \sqrt{(1-x)^2 - 4x^2}}{2x(1+x)}$.