

Date	Section	Exercises** (QC - Quick Check and CE - Class Exercises)
10/06*	6.1	QC - 1, 2; CE - 26, 27
10/08*	6.1	QC - 3; CE - 1, 2, 3, 5, 18, 28
10/10	6.1	CE - 30(b) - Here, distinct means labeled. , 33
10/13	Inversions	Inv. - 1, 2, 3
10/15	Inversions	Inv. - 4, 5
10/17	7.1	QC - 1; CE - 17, 19-21
10/22	7.1	QC - 2, 3; CE - 9, 10, 36
10/24	7.2	QC - 1; CE - 22-25
10/27*	8.1	CE - 9; Let $b_n = n^2/3^n$ and $B(x) = \sum_{n \geq 0} b_n x^n = (1 + x + x^2)^{-1}$. How should we refer to b_n within the context of this class? What about $B(x)$?
10/29*	8.1	See below.
10/31*	8.1	CE - 47; Also, see below.
11/03*	8.1	CE - 39, 40; Also, see below.
11/05*	8.1	See below.
11/07*	8.1	See below.

10/06 Show that if $\pi^2 = \text{id}$ (the identity permutation), then all cycles have length 1 or 2.

10/08

1. Show that for $n \geq 2$,

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k = 0$$

Solution:

Let $\mathfrak{S}_n = \mathcal{E} + \mathcal{O}$, where $\mathcal{E}$ and $\mathcal{O}$ contain permutations from $\mathfrak{S}_n$ with an even number of cycles and an odd number of cycles, resp. Also, since $n \geq 2$, the element 2 must appear in the first or second cycle of any $\pi \in \mathfrak{S}_n$.

Now define $\Gamma: \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ as follows. If 2 appears in the first cycle, say $\pi = (1a_1 \cdots a_j 2b_1 \cdots b_k)(c_1 \cdots) \cdots$, then define $\Gamma(\pi) = (1a_1 \cdots a_j)(2b_1 \cdots b_k)(c_1 \cdots) \cdots$. On the other hand, if 2 appears in the second cycle, say $\pi = (1a_1 \cdots a_j)(2b_1 \cdots b_k)(c_1 \cdots) \cdots$, then define $\Gamma(\pi) = (1a_1 \cdots a_j 2b_1 \cdots b_k)(c_1 \cdots) \cdots$. Notice that Γ is a well defined map that changes the parity (odd/even) on the cycle count of each permutation and that Γ^2 is the identity. It follows that Γ is a bijection with no fixed points. Hence, $\Gamma: \mathcal{E} \rightarrow \mathcal{O}$ is a bijection and $|\mathcal{E}| = |\mathcal{O}|$.

It follows that

$$\begin{aligned} \sum_k \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k &= \sum_{k \text{ even}} \begin{bmatrix} n \\ k \end{bmatrix} - \sum_{k \text{ odd}} \begin{bmatrix} n \\ k \end{bmatrix} \\ &= |\mathcal{E}| - |\mathcal{O}| \\ &= 0 \end{aligned}$$

2. Use a combinatorial argument to show that

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = \frac{n!}{2} \sum_{m=1}^{n-1} \frac{1}{m(n-m)}$$

** Exercises from the *A Walk Through Combinatorics*, 4th ed., Miklós Bóna, World Scientific

Solution:

Fix $1 \leq m \leq n-1$ and let (c_j) be a cycle, $j = 1, 2$. Also, let $\pi = (c_1)(c_2) \in \binom{[n]}{2}$ with $c_1 = a_1 a_2 \cdots a_m$ and $c_2 = b_1 b_2 \cdots b_{n-m}$. Notice that there are $\binom{n}{m}$ ways to choose the elements of (c_1) and there $(m-1)!$ distinct ways to arrange these elements. Finally, there are $(n-m-1)!$ ways to arrange the elements in (c_2) . So by the Product rule there are $\binom{n}{m}(m-1)!(n-m-1)!/2$ ways to produce a 2-cycle with exactly m elements in the first cycle. Since $\binom{n}{2}$ counts all 2-cycles in S_n we have

$$\begin{aligned} \binom{n}{2} &= \sum_{m=1}^{n-1} \binom{n}{m} \frac{(m-1)!(n-m-1)!}{2} \\ &= \frac{1}{2} \sum_{m=1}^{n-1} \frac{n!}{m!(n-m)!} (m-1)!(n-m-1)! \\ &= \frac{n!}{2} \sum_{m=1}^{n-1} \frac{1}{m(n-m)} \end{aligned}$$

as desired.

3. Prove that if $n \geq 0$ then

$$\sum_k (-1)^{n-k} \binom{n}{k} x^k = x^n \quad (1)$$

10/27 Find closed forms of the ordinary generating functions for each of the sequences below.

- (a) $\{(-1)^n\}_{n \geq 0} = \{1, -1, 1, -1, \dots\}$
 (b) $a_{n+1} = 3a_n + 1$, where $a_0 = 1$

10/29

1. Find the partial fraction decomposition of each of the rational functions below. In other words, find A and B in each case.

(a) $\frac{7x}{(2-3x)(1+2x)} = \frac{A}{2-3x} + \frac{B}{1+2x}$
 (b) $\frac{2x-6}{x^2-6x+4} = \frac{A}{x-3-\sqrt{5}} + \frac{B}{x-3+\sqrt{5}}.$

2. Suppose that $A(x) \xrightarrow{\text{ogf}} \{a_n\}_{n \geq 0}$ with sequence of coefficients as defined by the recursion below.

$$a_{n+2} = 5a_{n+1} - 6a_n, \quad a_0 = 2, \quad a_1 = 5$$

- (a) Generate the next five terms of this sequence.
 (b) Find the closed form of $A(x)$.

Note: As a check, you can replace $A(x)$ with your expression [here](#) and compare the coefficients of the result with your answer in part (a).

10/31

1. Find the closed form of the ordinary generating functions for each of the sequences below.

(a) $p_n = n$

- (b) $a_{n+1} = 2a_n + n$, $a_0 = 1$ *Hint:* Part (a) may help.
- (c) $b_{n+2} = b_{n+1} + 2b_n$, $b_0 = b_1 = 1$
2. Let $c_n = [x^n] \frac{7x}{(2-3x)(1+2x)}$. Find a closed formula for c_n . *Hint:* See Problem 1(a) from 10/29 above.

11/03

- Find a closed formula for each of the sequences defined below.
 - $b_{n+1} = 3b_n - 2^{n+1}$, $b_0 = 3$
 - $g_n = [x^n] \frac{1}{1-4x+3x^2}$ $n \geq 0$
 - $a_{n+2} = 5a_{n+1} - 6a_n$, $a_0 = 1, a_1 = 5$
- In how many ways can an n -day school year be created so that there is one holiday during the first part of the year, two holidays during the second part, and two field trips during the third part?
 - Same as part (a) except that the field trips are *distinguishable*. For example, students could visit an art museum during the first field trip and attend a sporting event during the second field trip, or students could first attend the sporting event followed by a visit to the art museum. These should be treated as two different semesters.
- Let B_n be a 1 by n board (one horizontal row) that consists of n squares. For $n > 0$, let b_n count the number of ways to cover B_n with a combination of 1 by 1 tiles (called *monominos*) and 1 by 2 tiles (called *dominos*) and let $b_0 = 1$. For example, the only way to cover B_1 is with one monomino, so $b_1 = 1$, but there are two ways to cover B_2 : We can use either 2 monominos or 1 domino, so $b_2 = 2$. *Note:* We can illustrate this as 1 + 1 and 2, respectively.
 - How many ways can you cover B_3 and B_4 using monominos and dominos?
 - Find a recursion equation satisfied by b_n .

11/05

- Let $f_n = |B_n|$ as described in class (or problem 3 on 11/03 above). Give direct combinatorial proofs of each of the following identities.
 - For $n \geq 1$, show that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n} - 1$.
 - For $n \geq 0$, show that $f_0 + f_2 + \cdots + f_{2n} = f_{2n+1}$.
 - For $n \geq 0$, show that $\sum_{k=1}^n \binom{n}{k} f_{k-1} = f_{2n-1}$. *Hint:* Condition on the number of monominos that appear among the first n tiles. And remember, a tile is either a monomino or a domino.
- In class we gave a combinatorial proof of the identity below. Answer the questions that follow.

$$\sum_{k=0}^n f_k = f_{n+2} - 1, \quad n \geq 0 \tag{2}$$

- Use generating functions to prove (2).
- Let $g_n = \sum_{k=0}^n f_k$. What “unexpected” recursion equation does g_n satisfy?

11/07

1. Let $f_n = |B_n|$ as described in class (or problem 3 on 11/03 above). Answer the questions below.
 - (a) Find the closed form of the generating function $G(x) = \sum_{n \geq 0} f_{2n+1} x^n$.
 - (b) For $n \geq 0$, show that $\sum_{k=0}^n (-1)^{n-k} f_{2k+1} = f_n f_{n+1}$.