1. Show that
$$x^n = \sum_{k=0}^{n} \binom{n}{k} (1+x)^k (-1)^{n-k} \tag{1}$$

2. Let n and k be integers. Let $\left\lfloor {[n] \atop k} \right\rfloor$ be the collection of all partitions of $[n]$ into k linearly ordered blocks. As usual, let $\left\lfloor {0 \atop 0} \right\rfloor = 1$ and for $n > 0$, let $\left\lfloor {n \atop k} \right\rfloor = \left| \left\lfloor {[n] \atop k} \right\rfloor \right|$. $\left\lfloor {n \atop k} \right\rfloor$ are called the Lah numbers (or Stirling Numbers of the 3rd kind). For example, $\left\lfloor {[3] \atop 2} \right\rfloor = \{12/3,\ 21/3,\ 13/2,\ 31/2,\ 23/1,\ 32/1\}$. It follows that $\left\lfloor {3 \atop 2} \right\rfloor = 6$. Notice that only the ordering within each block matters, not the order of the blocks themselves, so $32/1 = 1/32$, etc. It turns out that these numbers satisfy the following recursion.
$$\left\lfloor {n+1 \atop k} \right\rfloor = (n+k) \left\lfloor {n \atop k} \right\rfloor + \left\lfloor {n \atop k-1} \right\rfloor \tag{2}$$
together with additional boundary conditions $\left\lfloor {n \atop k} \right\rfloor = 0$ whenever $n < 0$ or $k \leq 0$ or $k > n$.

   (a) Find a combinatorial proof of the recursion (2).

   (b) Let $F(x) = \sum_{n \geq 0} \left\lfloor {n \atop k} \right\rfloor \frac{x^n}{n!}$. Show that
   $$F(x) = \frac{1}{k!} \left(\frac{x}{1-x} \right)^k \tag{3}$$
   and use (3) to show that
   $$\left\lfloor {n \atop k} \right\rfloor = \frac{n!}{k!} \binom{n-1}{k-1}$$

3. Count the number of nonnegative integer solutions for the equation below.
$$a + b + c + d = 10$$

4. Recall that we used the multi-choose coefficient for bins and beans arguments. Show that $\left(\!\binom{n}{k}\!\right)$ counts the sequences $\{a_j\}_{j=1}^{k}$ where $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq n$.

5. Find a closed form for the generating function of the sequence $0, 1, 4, 9, \ldots, n^2, \ldots$.

6. Recall the absorption/extraction property: $k\binom{n}{k} = n\binom{n-1}{k-1}$. Use this to show that for $n \geq 1$ we have
$$\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$$

7. If $m, n \in \mathbb{N}$ show that
$$\sum_{k=0}^{n} \binom{m+k}{k} = \binom{m+n+1}{n} \tag{4}$$

8. Use a combinatorial argument to count the number of different six-card hands that can be dealt from 3 combined standard card decks. Generalize for r combined decks.

9. For $n \geq 0$, show that

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x_{(k)}$$

10. Find a combinatorial proofs for the identities below. *Do not convert to binomial coefficients.*

   (a) For $n \geq 1$, $k \geq 0$,
   $$\left(\!\!\binom{n}{k}\!\!\right) = \left(\!\!\binom{k+1}{n-1}\!\!\right)$$

   (b) For $n, k \geq 0$ (except $n = k = 0$),
   $$\left(\!\!\binom{n}{k}\!\!\right) = \left(\!\!\binom{n}{k-1}\!\!\right) + \left(\!\!\binom{n-1}{k}\!\!\right)$$

   (c)
   $$k\left(\!\!\binom{n}{k}\!\!\right) = n\left(\!\!\binom{n+1}{k-1}\!\!\right)$$

11. Answer the following.

   (a) How many subsets of $[10] = \{1, 2, 3, \ldots, 10\}$ contain at least one odd number?

   (b) How many different ways can the letters of the word MISSISSIPPI be arranged if S's cannot appear consecutively?

   (c) In how many ways can 7 people be seated in a circle if two arrangements are considered the same whenever each person has the same neighbors (but not necessarily on the same sides)?

   (d) A group of 4 children from school A play with a group of 6 children from school B. In how many ways can children from different schools pair up (so at any one time, two of the children from school B will be left out)?

   (e) How many permutations π of [6] satisfy $\pi(1) \neq 2$?

12. Let $N(x) = x(1-x)^{-2}$ and notice that the counting sequence is $\{n\}_{n\geq 0}$.

   (a) Let $\sum_n f_n x^n = E(x) = (1 - N(x))^{-1} - 1$ and find the first 6 terms of $\{e_n\}_n$.

   (b) The sequence above is actually the even numbered terms of a very famous sequence. Identify the sequence and prove your claim.