1. Show that
$$x^n = \sum_{k=0}^{n} \binom{n}{k} (1+x)^k (-1)^{n-k} \tag{1}$$

Solution:

According to the Binomial Theorem
$$(1+x)^n = \sum_k \binom{n}{k} x^k$$

Now (1) follows by inversion.

We can also prove this directly. We have
$$\begin{aligned}\sum_k \binom{n}{k}(1+x)^k(-1)^{n-k} &\overset{*}{=} \sum_k \binom{n}{k}(-1)^{n-k}\sum_j \binom{k}{j}x^j \\ &= \sum_j x^j(-1)^n \sum_k \binom{n}{k}\binom{k}{j}(-1)^k \\ &\overset{**}{=} \sum_j x^j(-1)^n(-1)^n\delta_{nj} \\ &= x^n\end{aligned}$$

as expected. Notice that we used the Binomial Theorem at step (*) and Proposition 1 at step (**).

2. Let n and k be integers. Let $\left\lfloor {[n] \atop k} \right\rfloor$ be the collection of all partitions of $[n]$ into k linearly ordered blocks. As usual, let $\left\lfloor {0 \atop 0} \right\rfloor = 1$ and for $n > 0$, let $\left\lfloor {n \atop k} \right\rfloor = \left|\left\lfloor {[n] \atop k} \right\rfloor\right|$. $\left\lfloor {n \atop k} \right\rfloor$ are called the Lah numbers (or Stirling Numbers of the 3rd kind). For example, $\left\lfloor {[3] \atop 2} \right\rfloor = \{12/3,\ 21/3,\ 13/2,\ 31/2,\ 23/1,\ 32/1\}$. It follows that $\left\lfloor {3 \atop 2} \right\rfloor = 6$. Notice that only the ordering within each block matters, not the order of the blocks themselves, so $32/1 = 1/32$, etc. It turns out that these numbers satisfy the following recursion.
$$\left\lfloor {n+1 \atop k} \right\rfloor = (n+k)\left\lfloor {n \atop k} \right\rfloor + \left\lfloor {n \atop k-1} \right\rfloor \tag{2}$$
together with additional boundary conditions $\left\lfloor {n \atop k} \right\rfloor = 0$ whenever $n < 0$ or $k \leq 0$ or $k > n$.

   (a) Find a combinatorial proof of the recursion (2).

   Solution:

   Throughout this proof, a partition means a partition with linearly ordered blocks. The left-hand side counts the number of partitions of $[n+1]$ into k linearly ordered blocks.

   Now for any partition in $\left\lfloor {[n+1] \atop k} \right\rfloor$, $n+1$ is either alone in a block or it is not.
   In the first case, we can append $n+1$ to any of the partitions in $\left\lfloor {[n] \atop k-1} \right\rfloor$ to create a partition in $\left\lfloor {[n+1] \atop k} \right\rfloor$. Clearly there are $\left\lfloor {n \atop k-1} \right\rfloor$ ways to do this.

Otherwise, we can choose $\lambda \in \left\lfloor {[n] \atop k} \right\rfloor$, say $\lambda = B_1/B_2/\cdots/B_k$. Now we can place $n+1$ at the beginning of any block, e.g., $\lambda^j = B_1/B_2/\cdots/(n+1)B_j/\cdots/B_k$, so there are $k\left\lfloor {n \atop k} \right\rfloor$ ways to do this. Or we can place $n+1$ after any element (within any block), so there must be $n\left\lfloor {n \atop k} \right\rfloor$ ways to do this.

Since the 3 cases are distinct, we have shown that

$$\left\lfloor {n+1 \atop k} \right\rfloor = \left\lfloor {n \atop k-1} \right\rfloor + k\left\lfloor {n \atop k} \right\rfloor + n\left\lfloor {n \atop k} \right\rfloor$$

which is (2).

(b) Let $F_k(x) = \sum_{n\geq 0} \left\lfloor {n \atop k} \right\rfloor \frac{x^n}{n!}$. Show that

$$F_k(x) = \frac{1}{k!}\left(\frac{x}{1-x}\right)^k \tag{3}$$

and use (3) to show that

$$\left\lfloor {n \atop k} \right\rfloor = \frac{n!}{k!}\binom{n-1}{k-1} \tag{4}$$

Solution:

To show (4), we have

$$\begin{aligned} \left\lfloor {n \atop k} \right\rfloor = n![x^n]F_k(x) &= \frac{n!}{k!}[x^n]\left(\frac{x}{1-x}\right)^k \\ &= \frac{n!}{k!}[x^n]\left(\frac{x}{1-x}\right)^k \frac{1-x}{1-x} \\ &= \frac{n!}{k!}[x^n]\left(\frac{x^k}{(1-x)^{k+1}} - x\,\frac{x^k}{(1-x)^{k+1}}\right) \\ &= \frac{n!}{k!}\left(\binom{n}{k} - \binom{n-1}{k}\right) \\ &= \frac{n!}{k!}\binom{n-1}{k-1} \end{aligned}$$

3. Count the number of nonnegative integer solutions for the equation below.

$$a+b+c+d = 10$$

Solution:

This is simply a bins and beans argument with 4 bins (hence 3 bars) and 10 beans. For example, the solution $3+0+1+6 = 10$ can be represented as the ordered 4-tuple $(3, 0, 1, 6)$ or as

OOO||O|OOOOOO

using bins and beans. It follows that there are $\binom{13}{3} = 286$ such solution 4-tuples.

Using multi-choose terminology, we have 10 objects of 4 different kinds. That is, $\left(\!\binom{4}{10}\!\right)$. However, as we have seen,

$$\left(\!\!\binom{4}{10}\!\!\right) = \binom{10+4-1}{4-1} = \binom{10+4-1}{10}$$

as we saw above. Finally, we can relate this to the method we used to count the number of ways that one could purchase 10 items of fruit of 4 different kinds, say (a)pples, (b)ananas, (c)herrys and (d)urians. ☺

4. Recall that we used the multi-choose coefficient for bins and beans arguments. Show that $\left(\!\binom{n}{k}\!\right)$ counts the sequences $\{a_j\}_{j=1}^k$ where $1 \le a_1 \le a_2 \le \cdots \le a_k \le n$.

   Solution:

   How many ways can we choose k numbers (with repeats) from the set $[n]$? In other words, we wish to distribute k beans into n bins. Evidently, this is

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k} = \left(\!\binom{n}{k}\!\right)$$

5. Find a closed form for the generating function of the sequence $0, 1, 4, 9, \ldots, n^2, \ldots$.

   Solution:

   Let G be the generating function for the given sequence. Then

$$G(x) = \sum_{n\ge 0} n^2 x^n = \sum_{n \ge 1} n^2 x^n$$

   We suspect that this might be a derivative of a well-known power series, but the coefficients seem to be slightly off. However, notice that $n^2 = n(n+1) - n$. It follows that

$$\begin{aligned} G(x) &= \sum_{n\ge 1}(n+1)nx^n - \sum_{n\ge 1} nx^n \\ &= x\left[\sum_{n\ge 1}(n+1)nx^{n-1} - \sum_{n\ge 1} nx^{n-1}\right] \\ &= x\left[D_x^2\left(\frac{1}{1-x}\right) - D_x\left(\frac{1}{1-x}\right)\right] \\ &= \vdots \\ &= \frac{x+x^2}{(1-x)^3} \end{aligned}$$

6. Recall the absorption/extraction property: $k\binom{n}{k} = n\binom{n-1}{k-1}$. Use this to show that for $n \ge 1$ we have

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$$

   Solution:

This is straight forward.

$$\begin{aligned}\sum_{k=0}^{n} k\binom{n}{k} &= \sum_{k=1}^{n} k\binom{n}{k} \\ &= n\sum_{k=1}^{n}\binom{n-1}{k-1} \\ &= n\sum_{k=0}^{n-1}\binom{n-1}{k} \\ &= n2^{n-1}\end{aligned}$$

There is also a combinatorial argument. How many ways are there to choose a committee (of any size) from a class of n students, where one of the committee members is designated as chair? For example, in our own class (of 26 students) there are $\binom{26}{5}$ ways to choose a committee of 5 students and then 5 ways to designate a committee chair.

LHS: So for $0 \leq k \leq n$ there are $k\binom{n}{k}$ such choices. Summing over committees of any size from 0 to n, we obtain the left-hand side.

RHS: First select a chair from a class of n students. And there 2^{n-1} ways to choose a subset from the remaining $n-1$ students to form the rest of the committee. Now apply the Product Principle.

7. If $m, n \in \mathbb{N}$ show that
$$\sum_{k=0}^{n} \binom{m+k}{k} = \binom{m+n+1}{n} \tag{5}$$

Solution:

We induct on n. The identity certainly holds for $n = 0$. Now suppose that (5) holds for an arbitrary $n > 0$. We must then show
$$\sum_{k=0}^{n+1} \binom{m+k}{k} = \binom{m+n+2}{n+1}$$

Now
$$\begin{aligned}\sum_{k=0}^{n+1} \binom{m+k}{k} &= \sum_{k=0}^{n} \binom{m+k}{k} + \binom{m+n+1}{n+1} \\ &= \binom{m+n+1}{n} + \binom{m+n+1}{n+1} \\ &= \binom{m+n+2}{n+1}\end{aligned}$$

where the last line follows by the addition formula for binomial coefficients.

8. Use a combinatorial argument to count the number of different six-card hands that can be dealt from 3 combined standard card decks. Generalize for r combined decks.

Solution:

For 2 decks, this is just $\sum_k \binom{52}{6-k}\binom{6-k}{k}$ as we have seen before. For 3 decks, we must count by different hand patterns but we can exclude any of the patterns already counted above.

Pattern	Count
$abcxxx$	$\binom{52}{4}\binom{4}{1}\binom{1}{0}$
$abbxxx$	$\binom{52}{3}\binom{3}{2}\binom{2}{1}$
$aaaxxx$	$\binom{52}{2}\binom{2}{2}\binom{2}{2}$

For example, given the pattern $abbxxx$, there are $\binom{52}{3}$ ways to choose the 3 distinct cards followed by $\binom{3}{2}$ to pick which 2 cards are duplicated followed by $\binom{2}{1}$ way to pick which of the duplicated cards is triplicated(sp?). It follows that the number of different 6-card hands that can be dealt using 3 decks is
$$\binom{52}{4}\binom{4}{1}\binom{1}{0} + \binom{52}{3}\binom{3}{2}\binom{2}{1} + \binom{52}{2}\binom{2}{2}\binom{2}{2} + \sum_k \binom{52}{6-k}\binom{6-k}{k}$$

9. For $n \geq 0$, show that

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k$$

Solution:

Done in class.

10. Find a combinatorial proofs for the identities below. *Do not convert to binomial coefficients.*

(a) For $n \geq 1$, $k \geq 0$,

$$\left(\!\!\binom{n}{k}\!\!\right) = \left(\!\!\binom{k+1}{n-1}\!\!\right)$$

Solution:

LHS: By our definition, the left-hand side is the number of ways to arrange k stars with $n-1$ bars.

RHS: If we swap the stars and bars, we obtain $n-1$ stars and k bars. Of course, the number of arrangements should still be the same but we express this notationally as the right-hand side.

(b) For $n, k \geq 0$ (except $n = k = 0$),

$$\left(\!\!\binom{n}{k}\!\!\right) = \left(\!\!\binom{n}{k-1}\!\!\right) + \left(\!\!\binom{n-1}{k}\!\!\right)$$

Solution:

LHS: By definition, the left-hand side counts the number of ways can to choose k fruits from n different kinds.

RHS: Suppose we order the different kinds of fruits (for reference only). In any particular arrangement, the last (nth) fruit is either chosen or not. If the last fruit is chosen, then there are $\left(\!\binom{n}{k-1}\!\right)$ ways to select the remaining $k-1$ fruits.

On the other hand, if the last fruit is not chosen, then there are $\left(\!\binom{n-1}{k}\!\right)$ ways to choose k fruits from $n-1$ types. Now apply the Sum principle.

Observe that the identity also follows directly from the sum rule of binomial coefficients. We have

$$\begin{aligned}
((\binom{n}{k-1})) + ((\binom{n-1}{k})) &= \binom{n+k-2}{n-1} + \binom{n+k-2}{n-2} \\
&= \binom{n+k-1}{n-1} \\
&= ((\binom{n}{k}))
\end{aligned}$$

(c)

$$k\left(\!\!\binom{n}{k}\!\!\right) = n\left(\!\!\binom{n+1}{k-1}\!\!\right)$$

Solution:

There is a combinatorial argument for this, but I don't think it is very helpful. Just use the definition and apply the standard absorbtion/extraction identity.

11. Answer the following.

(a) How many subsets of $[10] = \{1, 2, 3, \ldots, 10\}$ contain at least one odd number?

Solution:

Including the empty set there are $2^5 = 32$ subsets that contain no odd numbers. Since there are $2^{10} = 1024$ subsets in total, there must be $1024 - 32$ subsets that contain at least one odd number.

(b) How many different ways can the letters of the word MISSISSIPPI be arranged if S's cannot appear consecutively?

Solution:

Including the ends, put spaces between each of the seven letters MIIIIPP. Now the 4 S's can be placed in any of the 8 spaces (with at most one per space). For example,

_ M _ I _ I _ P _ P _ I _ I _

Notice that for the above (fixed) permutation, there $\binom{8}{4}$ ways to place the S's in the spaces. Now there $\binom{7}{4,2,1}$ distinguishable ways to arrange the letters MIIIIPP. So by the Product Principle, there are

$$\binom{8}{4}\binom{7}{4,2,1} = 7350$$

distinguishable permutations that do not include consecutive S's.

(c) In how many ways can 7 people be seated in a circle if two arrangements are considered the same whenever each person has the same neighbors (but not necessarily on the same sides)?

Solution:

$\frac{1}{2}(7-1)!$

(d) A group of 4 children from school A play with a group of 6 children from school B. In how many ways can children from different schools pair up (so at any one time, two of the children from school B will be left out)?

Solution:

By the Product principle, this is simply $(6)_4 = 6 \times 5 \times 4 \times 3$. To see this, we focus on the children from school A. Notice that once the first child finds a playmate from school B, there are only 5 children from school B that the second child can play with, and so on.

(e) How many permutations π of $[6]$ satisfy $\pi(1) \neq 2$?

Solution:

Since there are 5! permutations with a 1 in the second slot, there must be $6! - 5! = 5 \times 5!$ such permutations.

12. Let $N(x) = x(1-x)^{-2}$ and notice that the counting sequence is $\{n\}_{n\geq 0}$.

(a) Let $\sum_n f_n x^n = E(x) = (1 - N(x))^{-1} - 1$ and find the first 6 terms of $\{e_n\}_n$.

Solution:

Using a CAS, the first 11 terms are 0, 1, 3, 8, 21, 55, 144, 377, 987, 2584, 6765.

(b) The sequence above is actually the even numbered terms of a very famous sequence. Identify the sequence and prove your claim.

Solution:

These look like the even terms from the (shifted) Fibonacci sequence, 0, 1, 1, 2, 3, 5, 8, 13, 21. So let $F(x) = x(1-x-x^2)^{-1}$ (the ordinary generating of the shifted Fibonacci sequence). Then the even part of $F(x)$ is

$$\begin{aligned}
\frac{F(x)+F(-x)}{2} &= \frac{1}{2}\left(\frac{x}{1-x-x^2} + \frac{-x}{1+x-x^2}\right) \\
&= \frac{1}{2}\left(\frac{x+x^2-x^3-x+x^2+x^3}{(1-x-x^2)(1+x-x^2)}\right) \\
&= \frac{x^2}{1-3x^2+x^4} \\
&= E(x^2)
\end{aligned}$$