1. Let $\{a_n\}_{n\geq 0}$ be given by the recursion below. Answer the questions that follow.

$$a_{n+2} = 3a_{n+1} + a_n + 1, \quad a_0 = 1, a_1 = 4$$

   (a) Find the next 3 terms in the sequence.

   (b) Find the closed form of the generating function $A(x) = \sum_{n\geq 0} a_n x^n$.

2. Recall that the English alphabet contains 21 consonants and the 5 vowels, *a,e,i,o,u*.

   (a) If repeated letters are forbidden, how many 8-letter strings contain exactly 5 vowels?

   (b) If repeated letters are allowed, how many 8-letter strings contain exactly 5 vowels? For example, unlike part (a), the string *bacaituo* is now permitted.

3. Let d_n be the n^{th} derangement number, that is, let $d_n = !n$. Give a combinatorial proof of the identity below.

$$d_{n+1} = nd_n + nd_{n-1},\ n \geq 0,\ d_{-1} = 0,\ d_0 = 1$$

   Hint: Recall that if $\pi \in S_n$ is a derangement, then π has no singleton cycles.

4. Find the number of nonnegative integer solutions to the equation below subject to the restrictions that follow.

$$x_1 + x_2 + x_3 = 28, \tag{1}$$

   where $0 \leq x_1 \leq 9$, $0 \leq x_2 \leq 8$, and $0 \leq x_3 \leq 17$. Recall that the number of *nonnegative* integer solutions to (1) is given by the multichoose coefficient $\left(\!\binom{3}{28}\!\right)$.

5. Let $c_0 = 1$ and for $n > 0$, let c_n count the number of nonnegative integer sequences $t_1, t_2, \ldots, t_n$ satisfying $t_1 + t_2 + \cdots + t_j \geq j$ and $\sum_{k=1}^{n} t_k = n$. For example, $c_3 = 5$ since the only such sequences are

$$111 \quad 120 \quad 210 \quad 201 \quad 300$$

   Show that $\{c_n\}_{n\geq 0}$ are the Catalan numbers.

6. Let $\left\{ {[n] \atop k} \right\}_r$ denote the collection of all set partitions on $[n]$ such that the numbers $1, 2, \ldots, r$ are in distinct subsets. As usual, we then let $\left\{ {n \atop k} \right\}_r = \left| \left\{ {[n] \atop k} \right\}_r \right|$. Equivalently, we can say that $\left\{ {[n] \atop k} \right\}_r$ is the collection of all set partitions on $[n]$ such that the numbers $1, 2, \ldots, r$ are in the minimal elements within their own block.

   Note: These are not the same as the modified Stirling numbers defined on Exam 1.

   (a) Show that if $n > r > 0$ we have

$$\left\{ {n \atop k} \right\}_r = k \left\{ {n-1 \atop k} \right\}_r + \left\{ {n-1 \atop k-1} \right\}_r \tag{2}$$

   (b) Show that if $n \geq r > 0$ we have

$$\left\{ {n \atop k} \right\}_r + (r-1) \left\{ {n-1 \atop k} \right\}_{r-1} = \left\{ {n \atop k} \right\}_{r-1} \tag{3}$$

(c) Notice that (2) agrees with usual recursion for Stirling Numbers of the Second Kind. The difference is in the boundary conditions:

$$\left\{ {n \atop k} \right\}_r = 0 \qquad\qquad n < r$$

$$\left\{ {n \atop k} \right\}_r = \delta_{k,r} \qquad\qquad n = r$$

Use this information to generate the first six nonzero rows of the Pascal-like triangle for $\left\{ {n \atop k} \right\}_2$.

(d) Show that

$$\left\{ {n \atop r} \right\}_r = r^{n-r}, \quad n \geq r$$