Chapter 3

There Are A Lot Of Them. Elementary Counting Problems

In the first two chapters, we have explained how to use the Pigeon-hole Principle and the method of mathematical induction to draw conclusions from certain numbers. However, *to find* those numbers is not always easy. It is high time that we learned some fundamental counting techniques.

3.1 Permutations

Let us assume that n people arrived at a dentist's office at the same time. The dentist will treat them one by one, so they must first decide the order in which they will be served. How many different orders are possible?

This problem, that is, arranging different objects linearly, is so omnipresent in combinatorics that we will have a name for both the arrangements and the number of arrangements. However, we are going to answer the question first.

Certainly, there are n choices for the person who will indulge in dental pleasures first. How many choices are there for the person who goes second? There are only $n-1$ choices as the person who went first will not go second, but everybody else can.

The crucial observation now is that for *each of the n choices* for the patient to be seen first, we have $n-1$ *choices for the patient who will be seen second*. Therefore, we have $n(n-1)$ ways to select these two patients. If you do not believe this, try it out with four patients, called A, B, C, and D, and you will see that there are indeed 12 ways the first two lucky patients can be chosen.

We can then proceed in a similar manner: we have $n-2$ choices for the patient to be seen third as the first two patients no longer need to be seen. Then we have $n-3$ choices for the patient to be seen fourth, and so on, two choices for the patient to be seen next-to-last, and only one choice, the remaining, frightened patient, to be seen last. Therefore, the number of orders in which the patients can sit down in the dentist's chair is $n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$.

Definition 3.1. The arrangement of different objects into a linear order using each object exactly once is called a *permutation* of these objects. The number $n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ of all permutations of n objects is called n *factorial*, and is denoted by $n!$.

So we have just proved the following basic theorem.

Theorem 3.2. *The number of all permutations of an n-element set is $n!$.*

We note that by convention, $0! = 1$. If you really want to know why we choose $0!$ to be 1, and not, say, 0, here is an answer. Let us assume that there are n people in a room and m people in another room. How many ways are there for people in the first room to form a line and people in the second room to form a line? The answer is, of course, $n! \cdot m!$ as any line in the first room is possible with any line in the second room. Now consider the special case of $n = 0$. Then people in the second room can still form $m!$ different lines. Therefore, if we want our answer, $n!m!$ to be correct in this singular case too, we must choose $0! = 1$. You will soon see that there are plenty of other situations that show that $0! = 1$ is the good definition.

The number $n!$ is extremely important in combinatorial enumeration, as you will see throughout this book. You may wonder how large this number is, in terms of n. This question can be answered at various levels of precision. All answers that are at least somewhat precise require advanced calculus. Here we will just mention, without proof that

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n . \tag{3.1}$$

The symbol $n! \sim z(n)$ sign means that $\lim_{n\to\infty} \frac{n!}{z(n)} = 1$. Relation (3.1) is called *Stirling's formula*, and we will use it in several later chapters.

Example 3.3. How many different flags can we construct using colors red, white, and green if all flags must consist of three horizontal stripes of different colors?

Solution. By Theorem 3.2, the answer is $3! = 3 \cdot 2 \cdot 1 = 6$. It is easy to convince ourselves that this is indeed correct by listing all six flags: RWG, RGW, WRG, WGR and GWR, and GRW.

The simplicity of the answer to the previous question was due to several factors: we used each of our objects exactly once, the order of the objects mattered, and the objects were all different. In the rest of this section we will study problems *without* one or more of these simplifying factors.

Example 3.4. A gardener has five red flowers, three yellow flowers and two white flowers to plant in a row. In how many different ways can she do that?

This problem differs from the previous one in only one aspect: the objects are *not all different*. The collection of the five red, three yellow, and two white flowers is often called a *multiset*. A linear order that contains all the elements of a multiset exactly once is called a *multiset permutation*.

How many permutations does our multiset have? We are going to answer this question by reducing it to the previous one, in which all objects were different. Assume our gardener plants her flowers in a row, in any of A different ways, then sticks labels (say numbers 1 through 5 for the red flowers, 1 through 3 for the yellow ones, and 1 through 2 for the white ones) to her flowers so that she can distinguish them. Now she has ten different flowers, and therefore the row of flowers she has just finished working on can look in 10! different ways. We have to tell how many of these arrangements differ *only* because of these labels.

The five red flowers could be given five different labels in 5! different ways. The three yellow flowers could be given three different labels in 3! different ways. The two white flowers could be given two different labels in 2! different ways. Moreover, the labeling of flowers of different colors can be done independently of each other. Therefore, the labeling of all ten flowers can be done in $5! \cdot 3! \cdot 2!$ different ways once the flowers are planted in any of A different ways. Therefore, $A \cdot 5! \cdot 3! \cdot 2! = 10!$, or, in other words,

$$A = \frac{10!}{5! \cdot 3! \cdot 2!} = 2520.$$

This argument can easily be generalized to a general theorem. However, we will need a greater level of abstraction in our notations to achieve that. This is because we will take general variables for the number of objects, but also for the number of *different kinds* of objects. In other words, instead of saying that we have five red flowers, three yellow flowers, and two white flowers, we will allow flowers of k different colors, and we will say that there are a_1 flowers of the first color, a_2 flowers of the second color, a_3 flowers of the third color, and so on. We complete the set of these conditions by saying that we have a_k flowers of color k (or a_k flowers of the kth color).

This is a long set of conditions, so some shorter way of expressing it would certainly make it less cumbersome. We will achieve this by saying that we have a_i flowers of color i, for all $i \in [k]$. Instead of saying that we plant our flowers in a line, we will often say that we *linearly order* our objects.

Now we are in a position to state our general theorem.

Theorem 3.5. *Let $n, k, a_1, a_2, \cdots, a_k$ be nonnegative integers satisfying $a_1 + a_2 + \cdots + a_k = n$. Consider a multiset of n objects, in which a_i objects are of type i, for all $i \in [k]$. Then the number of ways to linearly order these objects is*

$$\frac{n!}{a_1! \cdot a_2! \cdot \cdots \cdot a_k!}.$$

Proof. This is a generalization of Example 3.4, and the same idea of proof works here. The reader should work out the details. □

Quick Check

(1) How many ways are there to permute the elements of the set [7] so that an even number is in the first position?
(2) How many ways are there to permute elements of the multiset $\{1, 1, 2, 2, 3, 4, 5, 6\}$ so that the first and last elements are different?
(3) A garden has two rectangular flower beds. In the first bed, we will plant five different flowers in a row. In the second bed, we will plant six flowers in a row, so that there will be two flowers of each of three kinds. For which flower bed do we have more possibilities of proceeding?

3.2 Strings over a Finite Alphabet

Now we are going to study problems in which we are not simply arranging certain objects, knowing how many times we can use each object, but rather construct strings, or words, from a finite set of symbols, which we call a finite *alphabet*. We will *not* require that each symbol occur a specific number of times; though we may require that each symbol occur *at most once*.

Theorem 3.6. *The number of k-digit strings over an n-element alphabet is n^k.*

Proof. We can choose the first digit in n different ways. Then, we can choose the second digit in n different ways as well since we are allowed to use the same digit again (unlike in case of permutations). Similarly, we can choose the third, fourth, etc., kth element in n different ways. We can make all these choices independently from each other, so the total number of choices is n^k. $\square$

Example 3.7. The number of k-digit positive integers is $9 \cdot 10^{k-1}$.

Solution. There are two ways one can see this. From Theorem 3.6, we know that the number of k-digit strings that can be made up from the alphabet $\{0, 1, \cdots, 9\}$ is 10^k. However, not all these yield a k-digit positive integer. Indeed, those with first digit 0 do not. What is the number of these *bad* strings? Disregarding their first digit, these strings are $(k-1)$-digit strings over $\{0, 1, \cdots, 9\}$ with no restriction, so Theorem 3.6 shows that there are 10^{k-1} of them. Therefore, the number of k-digit strings that do not start with 0, in other words, the number of k-digit positive integers is $10^k - 10^{k-1} = 9 \cdot 10^{k-1}$ as claimed.

Alternatively, we could argue as follows. We have 9 choices for the first digit (everything but 0), and ten choices for each of the remaining $k-1$ digits. Therefore, the number of total choices is $9 \cdot 10 \cdot 10 \cdot \cdots \cdot 10 = 9 \cdot 10^{k-1}$, just as in the previous argument.

Before we discuss our next example, we mention a general technique in enumeration, the method of *bijections*. Let us assume that there are many men and many women in a huge ballroom. We do not know the number of men, but we know that the number of women is exactly 253. We think that the number of men is also 253, but we are not sure. What is a fast way to test this conjecture? We can ask the men and women to form man-woman pairs. If they succeed in doing this, that is, nobody is left without a match, and everyone has a match of the opposite gender, then we know that the number of men is 253 as well. If not, then there are two possibilities: if some man did not find a woman for himself, then the number of men is more than 253. If some woman did not find a man, then the number of men is less than 253.

This technique of matching two sets element-wise and then conclude (in case of success) that the sets are equinumerous is very often used in combinatorial enumeration. Let us put it in a more formal context.

Definition 3.8. Let X and Y be two finite sets, and let $f : X \to Y$ be a function so that

(1) if $f(a) = f(b)$, then $a = b$, and
(2) for all $y \in Y$ there is an $x \in X$ so that $f(x) = y$,

then we say that f is a *bijection* from X onto Y. Equivalently, f is a bijection if for all $y \in Y$, there exists a *unique* $x \in X$ so that $f(x) = y$.

In other words, a bijection matches the elements of X with the elements of Y, so that each element will have exactly one match.

The functions that have only one of the two defining properties of bijections also have their own names.

Definition 3.9. Let $f : X \to Y$ be a function. If f satisfies criterion (1) of Definition 3.8, then we say that f is *one-to-one* or *injective*, or is an *injection*. If f satisfies criterion (2) of Definition 3.8, then we say that f is *onto* or *surjective*, or is a *surjection*.

Proposition 3.10. *Let X and Y be two finite sets. If there exists a bijection f from X onto Y, then X and Y have the same number of elements.*

Proof. The bijection f matches elements of X to elements of Y, in other words it creates pairs with one element from X and one from Y in each pair. If f created m pairs, then both X and Y have m elements. $\square$

The advantages of the bijective method are significant. Instead of enumerating the elements of X, we can enumerate the elements of Y if that is easier. Then, we can find a bijection from X onto Y. Let us illustrate this by computing the number of all subsets of $[n]$ without resorting to induction.

Example 3.11. The number of all subsets of an n-element set is 2^n.

Solution. We construct a bijection from the set of all subsets of an n-element set into that of all n-digit strings over the binary alphabet $\{0, 1\}$. As this latter set has 2^n elements by Theorem 3.6, it will follow that so does the former.

To construct the bijection, let B be any subset of $[n]$. Now let $f(B)$ be the string whose ith digit is 1 if and only if $i \in B$ and 0 otherwise. This way $f(B)$ will indeed be an n-digit word over the binary alphabet. Moreover, it is clear that given any string s of length n containing digits equal to 0 and

1 only, we can find the unique subset $B \subseteq [n]$ for which $f(B) = s$. Indeed, B will precisely consist of the elements $i \in [n]$ so that the ith element of s is 1.

Example 3.12. A city has recently built ten intersections. Some of these will get traffic lights, and some of those that get traffic lights will also get a gas station. In how many different ways can this happen?

Solution. It is easy to construct a bijection from the set of all distributions of lights and gas stations onto that of ten-digit words over the alphabet A, B, C. Indeed, for each distribution of these objects, we define a word over $\{A, B, C\}$ as follows: if the ith intersection gets both a gas station and a traffic light, then let the ith digit of the word that we are constructing be A, if only a traffic light, then let the ith digit be B, and if neither, then let the ith digit be C.

Clearly, this is a bijection, for any ten-digit word can be obtained from exactly one distribution of gas stations and traffic lights this way. So the number we are looking for is, by Proposition 3.10, the number of all ten-digit words over a three-digit alphabet, that is, 3^{10}.

Theorem 3.13. *Let n and k be positive integers satisfying $n \geq k$. Then the number of k-digit strings over an n-element alphabet in which no letter is used more than once is*

$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

Proof. Indeed, we have n choices for the first digit, $n-1$ choices for the second digit, and so on, just as we did in the case of factorials. The only difference is that here we do not necessarily use all our n objects, we stop after choosing k of them. $\square$

The number $n(n-1)\cdots(n-k+1)$ is sometimes denoted $(n)_k$.

Example 3.14. A president must choose five politicians from a pool of 20 candidates to fill five different cabinet positions. In how many different ways can she do that?

Solution. We can directly apply Theorem 3.13. We have a 20-element alphabet (the politicians) and we need to count the number of 5-letter words with no repeated letters. Therefore, the answer is $(20)_5 = 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16$. If the candidates are all equally qualified, it may take a while...

Quick Check

(1) How many six-digit positive integers are there in which the first and last digits are the same?
(2) How many six-digit positive integers are there in which the first and last digits are of the same parity?
(3) How many functions $f : [n] \to [n]$ are there for which there exists exactly one $i \in [n]$ satisfying $f(i) = i$?

3.3 Choice Problems

At the national lottery drawings in Hungary, five numbers are selected at random from the set [90]. To win the main prize, one must guess all five numbers correctly. How many lottery tickets does one need in order to secure the main prize?

This problem is an example of the last and most interesting kind of elementary enumeration problems, called choice problems. In these problems, we have to choose certain *subsets* of a given set. We will often require that the subsets have a specific size. The important difference from the previous two sections is that the order of the elements of the subset will not matter; for example, $\{1, 43, 52, 8, 3\}$ and $\{52, 1, 8, 43, 3\}$ are identical as subsets of [90].

The number of k-element subsets of $[n]$ is of pivotal importance in enumerative combinatorics. Therefore, we have a symbol and name for this number.

Definition 3.15. The number of k-element subsets of $[n]$ is denoted $\binom{n}{k}$ and is read "n choose k".

The numbers $\binom{n}{k}$ are often called *binomial coefficients*, for reasons that will become clear in Chapter 4.

Theorem 3.16. *For all nonnegative integers $k \le n$, the equality*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n)_k}{k!}$$

holds.

Proof. To select a k-element subset of $[n]$, we first select a k-element string in which the digits are elements of $[n]$. By Theorem 3.6, we can do that in $n!/(n-k)!$ different ways. However, in these strings the order of the elements *does* matter. In fact, each k-element subset occurs $k!$ times among these strings as its elements can be permuted in $k!$ different ways. Therefore, the number of k-element subsets is $1/k!$ times the number of k-element strings, and the proof follows. $\square$

Therefore, if we want to be absolutely sure to win at the Hungarian lottery, we have to buy $\binom{90}{5} = \frac{90 \cdot 89 \cdot 88 \cdot 87 \cdot 86}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 43949268$ tickets. If you do that, make sure you fill them out right...

Definition 3.17. Let $S \subseteq [n]$. Then the *complement* of S, denoted S^c is the subset of $[n]$ that consists precisely of the elements that are not in S. In other words, S^c is the unique subset of $[n]$ that for all $i \in [n]$ satisfies the following statement: $i \in S^c$ if and only if $i \notin S$.

The following proposition summarizes some straightforward properties of the numbers $\binom{n}{k}$. We choose to announce these easy statements as a proposition since they will be used incessantly in the coming sections.

Proposition 3.18. *For all nonnegative integers $k \leq n$, the following hold.*

(1)

$$\binom{n}{k} = \binom{n}{n-k}.$$

(2)

$$\binom{n}{0} = \binom{n}{n} = 1.$$

Proof. (1) We set up a bijection f from the set of all k-element subsets of $[n]$ onto that of all $n-k$-element subsets of n. This f will be simplicity itself: it will map any given k-element subset $S \subseteq [n]$ into its complement S^c. Then for any $n-k$-element subset $T \subseteq [n]$, there is exactly one S so that $f(S) = T$, namely $S = T^c$. So f is indeed a bijection, proving that the number of k-element subsets of $[n]$ is the same as that of $n-k$-element subsets of $[n]$, which, by definition, means that $\binom{n}{k} = \binom{n}{n-k}$.

(2) The first equality is a special case of the claim of part 1, with $k = 0$. To see that $\binom{n}{0} = 1$, note that the only 0-element subset of $[n]$ is the empty set.

$\square$

We note in particular that $\binom{0}{0} = 1$, and that sometimes it is convenient to define $\binom{n}{k}$ even in the case when $n < k$. It goes without saying that in that case, we define $\binom{n}{k} = 0$ as no set has a subset that is larger than the set itself.

Example 3.19. A medical student has to work in a hospital for five days in January. However, he is not allowed to work two consecutive days in the hospital. In how many different ways can he choose the five days he will work in the hospital?

Solution. The difficulty here is to make sure that we do not choose two consecutive days. This can be assured by the following trick. Let a_1, a_2, a_3, a_4, a_5 be the dates of the five days of January that the student will spend in the hospital, in increasing order. Note that the requirement that there are no two consecutive numbers among the a_i, and $1 \leq a_i \leq 31$ for all i is equivalent to the requirement that $1 \leq a_1 < a_2 - 1 < a_3 - 2 < a_4 - 3 < a_5 - 4 \leq 27$. In other words, there is an obvious bijection between the set of 5-element subsets of [31] containing no two consecutive elements and the set of 5-element subsets of [27].

Instead of choosing the numbers a_i, we can choose the numbers $1 \leq a_1 < a_2 - 1 < a_3 - 2 < a_4 - 3 < a_5 - 4 \leq 27$, that is, we can simply choose a five-element subset of [27], and we know that there are $\binom{27}{5}$ ways to do that.

The trick we used here is also useful when instead of requiring that the chosen elements are far apart, we even allow them to be identical.

Example 3.20. Now let us assume that we play a lottery game where five numbers are drawn out of [90], but the numbers drawn are put back into the basket right after being selected. To win the jackpot, one must have played the same multiset of numbers as the one drawn (regardless of the order in which the numbers were drawn). How many lottery tickets do we have to buy to make sure that we win the jackpot?

Solution. We are going to apply the same trick as in the previous example, just backwards. We claim there is a bijection from the set of 5-element multisets

$$1 \leq b_1 \leq b_2 \leq b_3 \leq b_4 \leq b_5 \leq 90 \tag{3.2}$$

onto the set of 5-elements subsets of [94]. Indeed, such a bijection f is given by $f(b_1, b_2, b_3, b_4, b_5) = (b_1, b_2 + 1, b_3 + 2, b_4 + 3, b_5 + 4)$. It is obvious that the numbers b_i satisfy the requirements given by (3.2) if and only if

$f(b_1, b_2, b_3, b_4, b_5) = (b_1, b_2 + 1, b_3 + 2, b_4 + 3, b_5 + 4)$ is a subset of [94]. Therefore, we need to buy $\binom{94}{5}$ lottery tickets to secure a jackpot.

There is nothing magic about the numbers 90 and 5 here. In fact, the same argument can be repeated in a general setup, to yield the following Theorem.

Theorem 3.21. *The number of k-element multisets whose elements all belong to $[n]$ is*

$$\binom{n+k-1}{k}.$$

The following table summarizes our enumeration theorems proved in this chapter.

	parameters	formula
Permutations	n distinct objects	$n!$
	a_i objects of type i, $\sum a_i = n$	$\frac{n!}{a_1!a_2!\cdots a_k!}$
Lists	n distinct objects list of length k	$(n)_k = \frac{n!}{(n-k)!}$
	n distinct letters words of length k	n^k
Subsets	k-element subsets of $[n]$	$\binom{n}{k}$
	k-element multisets with elements from $[n]$	$\binom{n+k-1}{k}$

Quick Check

(1) A company has 20 male and 15 female employees. How many ways are there to form a committee consisting of four male and three female employees of the company?

(2) A professor wants to schedule a total of three hours of office hours for the next five days. In how many ways is that possible if the length of each office hour must be an integer (in hours)?
(3) In one lottery, we have to correctly pick five numbers out of ten in order to win, repetitions are not possible, and the order of the chosen numbers does not matter. In another lottery, we have to correctly pick four numbers out of ten, repetitions are possible, and the order of the chosen numbers does not matter. In which lottery do we have a higher chance to win?

Notes

One of the most difficult exercises of this chapter is Exercise 24. The first mathematician to prove the formula given in that exercise was probably P. A. MacMahon [35], in 1916. The proof presented here is due to the present author [17]. A high-level survey (using commutative algebra) of results concerning magic squares can be found in "Combinatorics and Commutative Algebra" [48] by Richard Stanley, while a survey intended for undergraduate and starting graduate students is presented in Chapter 9 of "Introduction to Enumerative and Analytic Combinatorics" [11] by the present author.

Exercises

(1) How many functions are there from $[n]$ to $[n]$ that are not one-to-one?
(2) Prove that the number of subsets of $[n]$ that have an odd number of elements is 2^{n-1}.
(3) A company has 20 employees, 12 males and eight females. How many ways are there to form a committee of 5 employees that contains at least one male and at least one female?
(4) A track and field championship has participants from 49 countries. The flag of each participating country consists of three horizontal stripes of different colors. However, no flag contains colors other than red, white, blue, and green. Is it true that there are three participating countries with identical flags?
(5) In countries that currently belong to a certain alliance, 17 languages are spoken by at least ten million people. For any two of these languages, the alliance employs an interpreter who can translate documents from one language to the other, and vice versa. One journalist has recently

noted that when the soon-to-be admitted countries bring the number of languages spoken by at least ten million people in the alliance to 22, more than a hundred new interpreters will be needed. Was she right? (No interpreter works two jobs.)

(6) How many five-digit positive integers are there with middle digit 6 that are divisible by three?

(7) How many five-digit positive integers are there that contain the digit 9 and are divisible by three?

(8) How many ways are there to list the digits $\{1, 2, 2, 3, 4, 5, 6\}$ so that identical digits are not in consecutive positions?

(9) How many ways are there to list the digits $\{1, 1, 2, 2, 3, 4, 5\}$ so that the two 1s *are* in consecutive positions?

(10) A cashier wants to work five days a week, but he wants to have at least one of Saturday and Sunday off. In how many ways can he choose the days he will work?

(11) A car dealership employs five salespeople. A salesperson receives a 100-dollar bonus for each car he or she sells. Yesterday the dealership sold seven cars. In how many different ways could this happen? (Let us consider two scenarios different if they result in different bonus payments.)

(12) A traveling agent has to visit four cities, each of them five times. In how many different ways can he do this if he is not allowed to start and finish in the same city?

(13) A college professor has been working for the same department for 30 years. He taught two courses in each semester. The department offers 15 different courses. Is it sure that there were at least two semesters when this professor had identical teaching programs? (A year has two semesters.)

(14) A restaurant offers five different soups, ten main courses, and six desserts. Joe decided to order at most one soup, at most one main course, and at most one dessert. In how many ways can he do this?

(15) A student in physics needs to spend five days in a laboratory during her last semester of studies. After each day in the lab, she needs to spend at least six days in her office to analyze the data before she can return to the lab. After the last day in the lab, she needs ten days to complete her report that is due at the end of the last day of the semester. In how many ways can she choose her lab days if we assume that the semester is 105 days long?

(16)(a) Three friends, having the nice names A, B, and C played a pingpong tournament each day of a given week. There were no ties at the end of the tournament. Prove that there were two days when the final ranking of the three people was the same.

(b) A fourth person, called D, joined the company of the mentioned three. These four friends played a tennis competition each day for five weeks. When the five weeks were over, one of them noticed that none of their one-day tournaments resulted in a tie at the first place, or in a tie at the last place. Is it true that there were two contests with the same final ranking of players?

(c) Now A, B and, C are playing a round-robin chess tournament each day starting January 1. Each player plays against each other player once playing the white pieces, and once playing the black pieces. The three friends agreed that they will stop when there will be two days with completely identical results. (That is, if on the earlier day, A beat B when playing the whites, but played a draw with him when playing the blacks, then, on the last day the friends play, A has to beat B when playing the whites, and has to play a draw with him when playing the blacks, and the same coinciding results must occur for the pair (B, C), and for the pair (A, C).)

When their left-out friend, D, heard about their plan, she said "are you sure you want to do this? You might be playing chess for two years!" Was she exaggerating?

(17) Let $k \geq 1$, and let $b_1, b_2, \cdots, b_k$ be positive integers with sum less than n, where n is a positive integer. Prove that then

$$b_1!b_2! \cdots b_k! < n!$$

holds. Can you make that statement stronger?

(18) How many 6-digit positive integers are there in which the sum of the digits is at most 51?

(19) How many ways are there to select an 11-member soccer team and a 5-member basketball team from a class of 30 students if

(a) nobody can be on two teams
(b) any number of students can be on both teams
(c) at most one student can be on both teams?

(20) On the island of Combinatoria, all cars have license plates consisting of six numerical digits only. A witness to a crime could only give a partial description of the getaway car. In particular, she noticed that the license plate was from Combinatoria, there was only one digit that occurred more than once, and that digit occurred three times. A

police officer estimated that this information will exclude more than 90 percent of all cars as suspects. Was his estimate correct?

(21) (+) A round robin chess tournament had $2n$ participants from two countries, n from each country. There were no two players with the same number of points at the end. Prove that there was at least one player who scored at least as many points against his compatriots as against the players of the other country. (In chess, a player gets one point for a win and one half of a point for a draw.)

(22) (+)

(a) At a round robin chess tournament, at least 3/4 of the games ended by a draw. Prove that there were two players who had the same final score.

(b) Now assume the tournament has been interrupted after t rounds, that is, after each player has finished t games. (We assume, for simplicity, that the number of players is even.) Is it still true that if at least 3/4 of the games played ended by a draw, then there were two players with the same total score?

(c) Prove that if the games of the tournament are played in a random order (there are no rounds; one player can finish many games before another player starts), and the tournament is interrupted at some point. Could it happen that three 3/4 of the finished games ended by a draw, but there were no two players with the same total score?

(d) Is there a constant $K < 1$ such that if we organize the tournament as in the preceding case, and we interrupt the tournament at a point when at least K of the finished games ended by a draw, then there will always be two players with the same total score?

(23) In how many different ways can we place eight identical rooks on a chess board so that no two of them attack each other?

(24) (++) A *magic square* is a square matrix with nonnegative integer entries in which all row sums and column sums are equal. Let $H_3(r)$ be the number of magic squares of size 3×3 in which each row and column have sum r. Prove that

$$H_3(r) = \binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4}, \tag{3.3}$$

where $H_3(r)$ is the number of 3×3 magic squares of line sum r. We will return to formula (3.3) in Chapter 11. The material covered in that chapter will allow us to give a simpler proof to this result.

(25) How many ways are there to select a subset $S \subseteq [15]$ so that S does not have two distinct elements a and b for which $a+b$ is divisible by three?

(26) How many permutations of the set $[n]$ are there in which no entry is larger than both of its neighbors? (We can assume that the condition is automatically satisfied for the leftmost and the rightmost entry.)

Supplementary Exercises

(27) (-) How many three-digit positive integers contain two (but not three) different digits?

(28) (-) How many ways are there to list the letters of the word ALABAMA?

(29) (-) How many subsets does $[n]$ have that contain exactly one of the elements 1 and 2?

(30) (-) How many subsets does $[n]$ have that contain at least one of the elements 1 and 2?

(31) (-) How many three-digit positive integers start and end with an even digit?

(32) How many four-digit positive integers are there in which all digits are different?

(33) How many four-digit positive integers are there that contain the digit 1?

(34) How many n-element subsets $S \subseteq [2n]$ are there so that there are no two elements x and y in S satisfying $x + y = 2n + 1$?

(35) How many subsets $S \subseteq [2n]$ are there (of any size) so that there are no two elements x and y in S satisfying $x + y = 2n + 1$?

(36) How many three-digit numbers are there in which the sum of the digits is even? (We do not allow the first digit to be zero.)

(37) In this exercise, the words *precede* does not mean *immediately precede.*

   (a) In how many ways can the elements of $[n]$ be permuted if 1 is to precede 2 and 3 is to precede 4?

   (b) In how many ways can the elements of $[n]$ be permuted if 1 is to precede both 2 and 3?

(38) In how many ways can the elements of $[n]$ be permuted so that the sum of every two consecutive elements in the permutation is odd?

(39) Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where the p_i are distinct primes, and the a_i are positive integers. How many positive divisors does n have?

(40)(a) Let $d(n)$ be the number of positive divisors of n. For what numbers n will $d(n)$ be a power of 2?

(b) Is it true that for all positive integers n, the inequality $d(n) \leq 1 + \log_2 n$ holds?

(41) A student needs to work five days in January. He does not want to work on more than one Sunday. In how many ways can he select his five working days? (Assume that in the year in question, January has five Sundays.)

(42) (+) A host invites n couples to a party. She wants to ask a subset of the $2n$ guests to give a speech, but she does not want to ask *both* members of any couple to give speeches. In how many ways can she proceed?

(43) We want to select as many subsets of $[n]$ as possible so that any two selected subsets have at least one element in common. What is the largest number of subsets we can select?

(44) We want to select an ordered pair (A, B) of subsets of $[n]$ so that $A \cap B \neq \emptyset$. In how many different ways can we do this?

(45) We want to select three subsets A, B, and C of $[n]$ so that $A \subseteq C$, $B \subseteq C$, and $A \cap B \neq \emptyset$. In how many different ways can we do this?

(46) A two-day mathematics conference has n participants. Some of the participants give a talk on Saturday, some others give a talk on Sunday. Nobody gives more than one talk, and there may be some people who do not give a talk at all. At the end of the conference, a few talks are selected to be included in a book. In how many different ways is this all possible if we assume that there is at least one talk selected for inclusion in the book?

(47) A group organizing a faculty-student tennis match must match four faculty volunteers to four of the 13 students who volunteered to be in the match. In how many ways can they do this?

(48) Let P be a convex n-gon in which no three diagonals intersect in one point. How many intersection points do the diagonals of P have?

(49) A student will study 26 hours in preparation for an exam. She will due this in the course of six consecutive days. On each of these days, she will study either four hours, or five hours, or six hours. In how many different ways is this possible?

(50) (+) Andy and Brenda play with dice. They throw four dice at the same time. If at least one of the four dice shows a six, then Andy wins, if not, then Brenda. Who has a greater chance of winning?

(51) (+) A store has n different products for sale. Each of them has a

different price that is at least one dollar, at most n dollars, and is a whole dollar. A customer only has the time to inspect k different products. After doing so, she buys the product that has the lowest price among the k products she inspected. Prove that on average, she will pay $\frac{n+1}{k+1}$ dollars.

(52) In how many ways can we place n non-attacking rooks on an $n \times n$ chess board?

(53) A class is attended by n sophomores, n juniors, and n seniors. In how many ways can these students form n groups of three people each if each group is to contain a sophomore, a junior, and a senior?

(54) The National Football League consists of 32 teams. These teams are first divided into two conferences, the American Conference and the National Conference, each of which consists of sixteen teams. Then each conference is divided into four divisions of four teams each. Each division has a distinct name. In how many ways can this be done?

(55) Answer the question of the previous exercise if there are two teams from New York City in the National Football League, and they cannot be assigned to the same conference.

(56) Let $P_3(r)$ be the number of 3×3 magic squares that are symmetric to their main diagonal. Prove that $P_3(r) \leq (r+1)^3$. (Magic squares are defined in Exercise 24.)

(57) How many $n \times n$ square matrices are there whose entries are 0 or 1 and in which each row and column has an even sum?

(58) How many ways are there for n people to sit around a circular table if two seating arrangements are considered identical if each person has the same left neighbor in them?

Solutions to Exercises

(1) The number of all functions from $[n]$ to $[n]$ is n^n by Theorem 3.6. Indeed, such a function f is defined by the array $(f(1), f(2), f(3), \cdots, f(n))$, and any entry in this array can be any element of $[n]$. If f is a one-to-one function, then the array $(f(1), f(2), f(3), \cdots, f(n))$ is a permutation of the elements $1, 2, \cdots, n$ as it contains each of them exactly once. So the number of one-to-one functions from $[n]$ to $[n]$ is $n!$, by Theorem 3.2. Therefore, the number of functions from $[n]$ to $[n]$ that are *not* one-to-one is $n^n - n!$.

Remark: Note that we were asked to compute the number of functions that were *not* one-to-one, and we obtained that number in an indirect way. We first computed the number of all functions from $[n]$ to $[n]$, then we computed the number of all functions from $[n]$ to $[n]$ that *were* one-to-one, and then we subtracted the second number from the first.

This technique of "number of good objects is equal to that of all objects minus that of bad objects" is very often used in combinatorial enumeration. Several exercises in this chapter can be solved this way.

(2) As in the proof of Example 3.11, we can bijectively encode all subsets of $[n]$ by 0-1 sequences consisting of n digits. If we want this sequence to contain an odd number of ones, then we can choose the first $n-1$ digits any way we want. The last digit can be used to make sure that the number of all ones is odd. That is, if there were an odd number of ones among the first $n-1$ digits, then the last digit has to be a zero, otherwise it has to be a one. Therefore, we make a choice $n-1$ times, and each time we have two possibilities. So the total number of possibilities is 2^{n-1}.

(3) There are $\binom{20}{5}$ ways to choose five people out of our twenty employees. However, $\binom{12}{5}$ of these choices will result in male-only committees, and $\binom{8}{5}$ will result in female-only committees. Therefore, the number of good choices is $\binom{20}{5} - \binom{12}{5} - \binom{8}{5}$.

(4) There are $4 \cdot 3 \cdot 2 = 24$ different 3-color flags that can be made from our four colors. As $2 \cdot 24 = 48 < 49$, it follows from the general version of the Pigeon-hole Principle that there are three identical flags among any 49 such flags.

(5) There are $\binom{17}{2} = \frac{17 \cdot 16}{2} = 136$ pairs that can be formed of the 17 languages currently spoken by at least ten million people in the alliance. When the number of these languages grows to 22, the number of pairs of languages will be $\binom{22}{2} = \frac{22 \cdot 21}{2} = 231$, so 95 new interpreters will be needed. Therefore, the journalist was wrong.

(6) It is well-known (see Exercise 34 of Chapter 2) that a positive integer is divisible by three if and only if the sum of its digits is divisible by three. Therefore, a five-digit a integer with middle digit six is divisible by three if and only if the four-digit integer obtained by deleting the middle digit of a is divisible by three. There are 9000 four-digit positive integers, and the third, sixth, ninth,....,9000th of them are divisible by 3 (these are the integers 1002, 1005, 1008,...,9999). In other words, there are 3000 four-digit positive integers divisible by

three, so there are 3000 five-digit positive integers divisible by three and having middle digit 6.

(7) The number of all five-digit positive integers is 90000, and one third of them, 30000, are divisible by three. Let us count how many of these 30000 numbers do not contain the digit nine. Such a number can start with one of eight digits $(1, 2, \cdots, 8)$, then can have any of nine digits $(0, 1, 2, \cdots, 8)$ in the second, third, and fourth positions. For the fifth digit, we have more limited choice. We have to pick the fifth digit so that the sum of all five digits is divisible by three. Depending on the first four digits, we can either choose one of 0,3,6, or one of 1,4,7, or one of 2,5,8. Either way, this means three choices. The total number of choices we have is $8 \cdot 9^3 \cdot 3 = 17496$, so this is the number of 5-digit positive integers that are divisible by three, but do not contain the digit 9. Therefore, there are $30000 - 17496 = 12504$ 5-digit positive integers that are divisible by three and do contain the digit 9.

(8) The number of all permutations of this multiset is given by Theorem 3.5, and is equal to $\frac{7!}{2!} = 2520$. However, we have to subtract the number of those permutations in which the two identical digits are in consecutive positions. To count these, let us *glue* the two identical digits together. Then we have six digits, which are all different, and therefore Theorem 3.2 shows that they have $6! = 720$ permutations. Therefore, the number of all permutations of our multiset in which the two identical digits are not in consecutive positions is $2520 - 720 = 1800$.

(9) Just as in Exercise 8, let us glue the two 1s together. Then we simply have to count permutations of the multiset $\{1, 2, 2, 3, 4, 5\}$. Theorem 3.5 shows that there are $\frac{6!}{2!} = 360$ such permutations.

(10) There are $\binom{7}{5} = \binom{7}{2} = 21$ ways to choose five days of the week. Let us now count the *bad* choices, that is, those that contain both Saturday and Sunday. Clearly, there are $\binom{5}{3} = 10$ of these. Indeed, they contain Saturday, Sunday, and three of the remaining five days. Therefore, the number of good choices is $21 - 10 = 11$.

(11) As we only consider two scenarios different if they result in different bonus payments, we are not interested in the *order* in which the different salespeople sold the seven cars. What matters is how many cars each of them sold. Therefore, we are interested in the number of 7-element multisets whose elements are from the set [5]. By Theorem 3.21, this number is $\binom{5+7-1}{7} = \binom{11}{7} = \binom{11}{4} = 330$.

(12) There are $\frac{20!}{5!\cdot5!\cdot5!\cdot5!}$ ways to visit four cities, each of them five times.

Let us determine the number of ways to do this so that we start in city A, and end in city A. In that case, we are free to choose the order in which we make the remaining 18 visits. As three of those visits will be to city A, and five will be to each of the remaining three cities, this can be done in $\frac{18!}{5!\cdot 5!\cdot 5!\cdot 3!}$ ways. Obviously, the same argument applies for the number of visiting arrangements that start and end in B, that start and end in C, and that start and end in D. So the final answer is

$$\frac{20!}{5!\cdot 5!\cdot 5!\cdot 5!} - 4\cdot \frac{18!}{5!\cdot 5!\cdot 5!\cdot 3!}.$$

(13) No, that is not sure. There are $\binom{15}{2} = \frac{15\cdot 14}{2} = 105$ ways to pick two courses out of 15 courses, and 30 years consist of 60 semesters only.

(14) Joe can make one of six choices on soup as he may decide not to order soup at all. Similarly, he can make one of 11 choices on the main course, and one of seven choices on dessert. So the total number of possibilities is $6\cdot 11\cdot 7 = 462$.

(15) Let us number the days of the semester from 1 to 105, and let us denote the days when the student is in the lab by $a_1, a_2, \ldots, a_5$. Then the conditions imply that $a_5 \leq 95$, and

$$1 \leq a_1 < a_2 - 6 < a_3 - 12 < a_4 - 18 < a_5 - 24 \leq 95 - 24 = 71.$$

Denote $b_1 = a_1$, $b_2 = a_2 - 6$, $b_3 = a_3 - 12$, $b_4 = a_4 - 18$, and $b_5 = a_5 - 24$. Clearly, knowing the numbers b_i is equivalent to knowing the numbers a_i.

Note that $b_5 \leq 95 - 24 = 71$. There is no additional requirement for the numbers b_i besides $1 \leq b_1 < b_2 < b_3 < b_4 < b_5$, so there are $\binom{71}{5}$ possible choices for the set of these numbers. Therefore, our student can make this many choices.

(16)(a) There are $3! = 6$ ways the contest could end, and there are seven days in a week. We know, if from nowhere else, then from the title of Chapter 1, that Seven Is More Than Six. Therefore, the pigeon-hole principle implies that there were two contests with identical results.

(b) If there were no ties at all, the contest could end in $4! = 24$ different ways. If there is a tie, it could only be at the second-third place. The two people who tie can be chosen in $\binom{4}{2} = 6$ ways, then the winner can be either of the remaining two people. So there are $6\cdot 2 = 12$ different outcomes with a tie. Therefore the total number of possible endings for the competition is $24 + 12 = 36$. There are only 35 days in five weeks, so it is possible that there are no two days when the contest ends the same way.

(c) Each tournament consists of six games as we have three choices for the person leading the white pieces, and two choices leading the black pieces. Each of these six games can have three different results: either white wins, or black wins, or it is a draw. So there are $3^6 = 729$ ways the games of a tournament can end. Therefore, the three friends will play for at most 730 days, which is exactly two years as neither 2013, nor 2014 is a leap-year. So D was in fact right, she was not exaggerating.

(17) Let b_{k+1} be a positive integer so that $n = \sum_{i=1}^{k+1} b_i$. Theorem 3.5 then tells us that

$$T = \frac{n!}{b_1!b_2!\cdots b_{k+1}!}$$

is the number of linear orderings of n objects of $k+1$ various kinds, so that b_i objects are of kind i. In particular, $T = \frac{n!}{b_1!b_2!\cdots b_{k+1}!}$ is a positive integer, (as it is the number of elements in a nonempty set), so

$$\frac{n!}{b_1!b_2!\cdots b_k!} = b_{k+1}!T.$$

The right-hand side (and therefore, the left-hand side) is larger than 1 as long as one of T and b_{k+1} is larger than 1. The only way in which $T = 1$ could hold would be if there were no two distinct objects at all, but that is not possible since there is at least one object of type $k+1$, and one other object. So we proved that not only $b_1!b_2!\cdots b_k! < n!$, but also, $b_1 b_2 \cdots b_k$ is a proper divisor of $n!$.

(18) The number of all 6-digit integers is 900000 by Example 3.7. Again, we are going to count those which do *not* satisfy the criteria, that is, those with digit sum of at least 52. There are only four 6-element multisets of digits that sum to at least 52, namely $\{9, 9, 9, 9, 9, 9\}$, $\{9, 9, 9, 9, 9, 8\}$, $\{9, 9, 9, 9, 9, 7\}$, and $\{9, 9, 9, 9, 8, 8\}$. Theorem 3.5 implies that they have 1,6,6, and 15 multiset permutations (respectively), so altogether there are 28 numbers out of 900000 that violate the criteria. So the number of 6-digit positive integers that satisfy the criteria is 899972.

(19)(a) We have $\binom{30}{11}$ choices for the soccer team. Then we have to choose from the remaining 19 people in $\binom{19}{5}$ ways for the basketball team. Consequently, the final answer is $\binom{30}{11} \cdot \binom{19}{5}$.

(b) If there is no restriction at all, then after choosing the soccer team, we can choose the basketball team in $\binom{30}{5}$ ways, from the set of all students. So the total number of choices is $\binom{30}{11} \cdot \binom{30}{5}$.

(c) All $\binom{30}{11} \cdot \binom{19}{5}$ team compositions (computed in the first part in this exercise) in which no student is on two teams are certainly good. Apart from these, there are those in which there is exactly one student on both teams. We have 30 choices for this person, then there are $\binom{29}{10} \cdot \binom{19}{4}$ ways to choose the remaining players from the rest of the class. Thus the total number of possibilities is

$$\binom{30}{11} \cdot \binom{19}{5} + 30 \cdot \binom{29}{10} \cdot \binom{19}{4}.$$

(20) The digit that occurred three times could be any of ten digits. The positions of its three occurrences could be any of the $\binom{6}{3} = 20$ three-element subsets of $[6]$. The other three digits form a 3-digit word over the remaining 9-letter alphabet without repetition, so we have $9 \cdot 8 \cdot 7 = 504$ choices for them. As all these choices can be made independently from each other, the total number of our choices is $10 \cdot 20 \cdot 504 = 100800$. This is slightly more than ten percent of all license plates, which would be 100000, so the police officer was a little bit too optimistic.

(21) Let A be the country whose players scored, in totality, at most as many points in the international games as players from the other country. Take the n players from A, and let $a_1, a_2, \cdots, a_n$ denote the number of points they accumulated against their countrymen. Let $b_1, b_2, \cdots, b_n$ be the number of points they accumulated against players from country B. Now assume that our claim is false, that is, $a_i < b_i$ for all i. In other words, $a_i \leq b_i - 0.5$ for all i. Summing these inequalities over all $i \in [n]$, we get that

$$\sum_{i=1}^{n} a_i \leq \left(\sum_{i=1}^{n} b_i \right) - n/2. \tag{3.4}$$

On the other hand, note that $\sum_{i=1}^{n} a_i = n(n-1)/2$ as any two players from A played each other once, and in each of those games, one point was up for grabs. Comparing this with (3.4), we get

$$\frac{n(n-1)}{2} + \frac{n}{2} = \frac{n^2}{2} \leq \left(\sum_{i=1}^{n} b_i \right). \tag{3.5}$$

Similarly,

$$\sum_{i=1}^{n} b_i \leq n^2/2 \tag{3.6}$$

as players from A got at most half of all points that were available at the international games.

Comparing (3.5) and (3.6) we see that $\sum_{i=1}^{n} b_i = n^2/2$ must hold. That is, $\sum_{i=1}^{n} b_i$ is *exactly* $n/2$ larger than $\sum_{i=1}^{n} a_i$. Therefore, equality holds in (3.4), and so equality must hold in all equations of the type $a_i \leq b_i - 0.5$. (Recall that (3.4) was obtained by taking the sum of these equations for all i.) Therefore, for all i, we must have $a_i = b_i - 0.5$, meaning that the total score of the ith player from country A was $a_i + b_i = 2a + 0.5$, which is never an integer. Therefore, no player from country A has a final score that is an integer. By the very same argument, no player from country B has a final score that is an integer. Indeed, in totality, players from B scored $n^2/2$ points against players from A, so the same argument works.

This is a contradiction as we know there are no two players with the same final score. The number of possible non-integer final scores is less than $2n$: indeed, they are $0.5, 1.5, 2.5, \cdots (2n-1) - 0.5$, which is only $2n - 1$ different scores for the $2n$ players. So there must be a player who did better against his compatriots than against players from the other country.

(22)(a) Let us change the scoring system of chess as follows: a player gets one point for a win, zero points for a draw, and -1 points for a loss. Clearly, this does not change the facts in our problem: people who had different scores in the original scoring system have different scores now, and people who had identical scores in the original scoring system have identical scores now. Indeed, if player x won a_x games, got a draw b_x times, and lost c_x times, then his total score in the old system is $a_x + (b_x/2)$, and his total score in the new system is $a_x - c_x$. Assume player y got the same total score in the old system. That means

$$a_x + \frac{b_x}{2} = a_y + \frac{b_x}{2}.$$

Multiply this equation by 2, and subtract the equation $a_x + b_x + c_x = a_y + b_y + c_y$ from it. (The latter simply shows that both players played the same number of games.) We get

$$a_x - c_x = a_y - c_y,$$

which shows that the two players had the same score in the new system, too.

Let us assume that all n players had different final scores. Let $k = n/2$ if n is even, and let $k = (n-1)/2$ if n is odd. Then we can assume without loss of generality that there are k players with positive final scores. As these scores are all different, their sum is at least $1+2+\cdots+k = k(k+1)/2$. As only wins result in positive scores, there had to be at least $k(k+1)/2$ wins at the tournament. The number of all games is, on the other hand, $\binom{n}{2}$. Therefore, the ratio of wins (games not ended in a draw) and all games is

$$\frac{k(k+1)}{(n-1)n} > \frac{1}{4}. \tag{3.7}$$

(b) Yes, the same argument will work, except that the total number of games played will be less than $\binom{n}{2}$, therefore the denominator in formula (3.7) will decrease, therefore the ratio of wins will be even larger.

(c) The problem with the previous argument here is that if not all players complete the same number of games, then the new scoring system is *not* the same as the classical one. Indeed, the argument of part (a) would not work here as $a_x + b_x + c_x = a_y + b_y + c_y$ would not hold. The statement is no longer true. A counterexample can then be found for $n = 4$ as follows. Let games $A - B$, $A - C$ end by draws, and let game $B - D$ be won by B. Then B has 1.5 points, A has 1, C has 0.5, and D has 0. (Note that in the $1 - 0 - (-1)$ scoring system, A and C would both have 0 points.)

(d) No. Our counterexample will be a generalization of the preceding example, and also, of Example 1.7 of Chapter 1. Say we have n players, (n is even) $A_1, A_2, \cdots, A_{n-1}$ and B. Let A_{n-1} play with everyone, except for A_1, let A_{n-2} play with everyone except for A_1 and A_2, in general, let A_i play with A_j if $i+j > n$, and let A_i play with B if $i > n/2$. Let all these games end by a draw. Then A_i has $i/2$ points for all i, and B has $\frac{n}{4} - \frac{1}{2}$ points. The only problem now is that B has the same number of points as one of the players A_i. To correct that, let B play with all the A_i he did not (there are $\frac{n}{2}$ of those), and defeat them all. Then B becomes a clear winner of the tournament, and the points of the A_i do not change, so they stay all different. Also note that the number of games played is quadratic in n, whereas that of wins is linear in n, proving that the ratio of draws can be arbitrarily close to 1 if n is large enough.

(23) **First Solution.** We can place the first rook anywhere on the board, that is, we have $8^2 = 64$ choices for its position. The second rook cannot be in the row or column of the first one, leaving $7^2 = 49$ choices for its position. Similarly, we will have $6^2 = 36$ choices for

the position of the third rook, and so on. Therefore, if our rooks were distinguishable, we would have $8^2 \cdot 7^2 \cdots 1^2 = 8!^2$ ways to place them. However, they are indistinguishable, so it does not matter *which rook is in which position* as long as the *set of all rooks* covers the same eight positions. Consequently, we have counted every placement $n!$ times, and the number of all placements is $8!^2/8! = 8! = 40320$.

Second solution. Each $f : [8] \to [8]$ can be bijectively associated to a non-attacking rook placement as follows. For all $i \in [8]$, put a rook into the square $(i, f(i))$. This ensures that there will be exactly one rook in each row and column. It is also easy to see that this is a bijection, that is, all rook placements define one one-to-one function from [8] onto itself. So the number of rook placements is $n!$ by Exercise 1.

(24) Take any magic square of line sum r and side length 3. It is clear that the four elements shown in the figure determine all the rest of the square.

a	d	
	b	
		c

Indeed, the next table shows our only possible choice for each remaining entry. Thus all we need to do is to compute the number of ways we can choose a, b, c and d so that we indeed have that one choice, i.e., the obtained entries of the magic square are all nonnegative.

a	d	$r-a-d$
$r+c-$ $(a+d+b)$	b	$a+d-c$
$b+d-c$	$r-b-d$	c

The previous table shows that the entries of our matrix will be nonnegative if and only if the following inequalities hold:

$$a+d \leq r \tag{3.8}$$

$$b+d \leq r \tag{3.9}$$

$$c \leq a+d \tag{3.10}$$

$$c \leq b+d \tag{3.11}$$

$$a+d+b-c \leq r. \tag{3.12}$$

We will consider three different cases, according to the position of the smallest element on the main diagonal. In each of them, at least three of the five conditions above will become redundant, and we will only need to deal with the remaining one or two.

(a) Suppose $0 \leq a \leq b$ and $0 \leq a \leq c$. In this case conditions (3.8), (3.11), and (3.12) are clearly redundant, because they are implied by (3.9) and (3.10).
The crucial observation is that in all the three cases we can collect all our conditions into one single chain of inequalities. In this case we do it as follows:

$$a \leq 2a+d-c \leq a+b+d-c \leq b+d \leq r. \tag{3.13}$$

Indeed, the first inequality is equivalent to (3.10), the second one is equivalent to our assumption $a \leq b$, the third one is equivalent to our assumption $a \leq c$, and the last one is equivalent to (3.9). Moreover, note that once we know the terms of this chain, that is, $a, 2a+d-c, a+b+d-c$ and $b+d$, then we know a, b, c and d, too, thus we have determined the magic square. Thus all we need to do is simply count how many ways there are to choose these four terms. Inequality (3.13) shows that these terms are nondecreasing, therefore the number of ways to choose them is simply the number of 4-combinations of $r+1$ elements with repetitions allowed, which is $\binom{r+4}{4}$. (Recall that 0 is allowed to be an entry.)

(b) Now suppose $a > b$ and $c \geq b$. Then (3.9), (3.11) and (3.12) are redundant. Consider the chain of inequalities

$$b \leq 2b+d-c \leq a+b+d-c-1 \leq a+d-1 \leq r-1. \quad (3.14)$$

We can use the argument of the previous case to prove that (3.14) equivalent to (3.8), (3.12) and our assumptions, as the roles of a and b are completely symmetric. The only change is that here we do not count those magic squares in which $a = b$, and this explains the (-1) in the last three terms. Thus here we have to choose four elements in non-decreasing order out of the set $\{0, 1, \cdots, r-1\}$, which can be done in $\binom{r+3}{4}$ ways.

(c) Finally, suppose that $a > c$ and $b > c$. Then (3.8), (3.9), (3.10) and (3.11) are redundant. Condition (3.12) and our assumptions can be collected into the following chain:

$$c \leq b-1 \leq b+d-1 \leq a+b+d-c-2 \leq r-2. \quad (3.15)$$

Here the first inequality is equivalent to our assumption $c < b$, the second one says that d is nonnegative, the third one is equivalent to our assumption $c < a$, and the last one is equivalent to (3.12). The four terms of (3.15) determine a, b, c and d, and they can be chosen in $\binom{r+2}{4}$ ways, which completes the proof.

Thus the number of 3×3 magic squares of line sum r is indeed $\binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4}$. Furthermore, the three terms in this sum count the magic squares in which the (first) minimal element of the main diagonal is the first, second, or third element.

(25) The set [15] has five elements divisible by three, five elements of the form $3k+1$, and five elements of the form $3k+2$. Let S_0, S_1, and S_2 denote the sets of these elements. Then we can select one element

or no element of S_0, which gives us six possibilities. After this, we can either not select any more elements, or we can select a non-empty subset of S_1 in $2^5 - 1 = 31$ ways, or we can select a non-empty subset of S_2 in 31 ways. So the total number of choices we have for selecting elements from S_1 and S_2 is $1 + 31 + 31 = 63$, leading to the final count of $6 \cdot 63 = 378$.

(26) In such permutations, if an increasing subsequence starts, it must last till the end. So, a permutation satisfies this requirement if and only if it decreases all the way to its entry 1, then it increases all the way to the end, like 421356. This means that the set S of entries on the left of 1 completely determines the permutation. As S can be any subset of $\{2, 3, \cdots, n\}$, there are 2^{n-1} possibilities for S, so that is the number of permutations with the desired property.