Limit Theorems for Sequences

Convergent Sequences

A sequence $\{a_n\}$ is **bounded** if there is a real number M such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem Convergent sequences are bounded.

Proof: Let $\{a_n\}$ be a convergent sequence with limit s and let $\varepsilon = 623$. Then there exists a natural number N such that

$$n > N \quad \text{implies} \quad |s_n - s| < 623 \tag{1}$$

Thus

$$|s_n| = |s_n - s + s| \leq |s_n - s| + |s| \leq 623 + |s|$$

for all $n > N$.

Now let M be the maximum of the finite set

$$\{623 + |s|, |a_1|, |a_2|, \ldots, |a_N|\}.$$

Then $|a_n| \leq M$ for all $n \in \mathbb{N}$, as desired.

Note: The last proof makes use of a very important idea:

> All but a finite number of terms in a convergent sequence are arbitrarily close to the limit.

We will exploit this idea again below.

Proposition

Suppose that $\lim_{n\to\infty} b_n = b \neq 0$. Then there is a natural number N such that for all $n > N$, $b_n \neq 0$.

Proof: Without loss of generality we may assume $b > 0$. Now let $\varepsilon = \frac{b}{2} > 0$. So there is an $N \in \mathbb{N}$ such that for all $n > N$, $|b_n - b| < \frac{b}{2}$. Rearranging we see that this implies $b_n > \frac{b}{2} > 0$, as desired.

Note: The proof is similar if $b < 0$. This proposition is used in the next theorem.

The Limit Laws

Theorem

Suppose that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ and let k be a real number. Then

(a) $\lim_{n\to\infty} ka_n = ka$

(b) $\lim_{n\to\infty}(a_n + b_n) = a + b$

(c) $\lim_{n\to\infty} (a_n b_n) = ab$

(d) $\lim_{n\to\infty} \left(\dfrac{a_n}{b_n}\right) = \dfrac{a}{b}, \quad b \neq 0,\ b_n \neq 0$ for all $n \in \mathbb{N}$

Before proving (d), let's look at an example.

Example: Show that $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} a_n^2 = 0$.

Proof: If $\lim_{n\to\infty} a_n = L \neq 0$, then by (c), $\lim_{n\to\infty} a_n^2 = L^2 \neq 0$. This establishs the right to left implication. We leave the forward implication as an easy exercise.

The Limit Laws (cont)

To prove property (d), we first note that by the previous proposition, there exists $N_1 \in \mathbb{N}$, such that for all $n > N_1$, $|b_n| > \frac{|b|}{2}$. Now let $\varepsilon > 0$. There exists $N_2, N_3 \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\varepsilon|b|}{4}, \quad \text{provided } n > N_2$$
$$|b_n - b| < \frac{\varepsilon b^2}{4|a|}, \quad \text{provided } n > N_3$$

Now let $n > N = \max\{N_1, N_2, N_3\}$, then

$$\begin{aligned}\left|\frac{a_n}{b_n} - \frac{a}{b}\right| &= \frac{|a_n b - a b_n|}{|b_n b|} = \frac{|a_n b - ab + ab - a b_n|}{|b_n b|} \\ &\leq \frac{1}{|b_n b|}(|b||a_n - a| + |a||b - b_n|) \\ &< \frac{2}{b^2}(|b||a_n - a| + |a||b - b_n|) \\ &< \frac{2|b|}{b^2}\left(\frac{\varepsilon|b|}{4}\right) + \frac{2|a|}{b^2}\left(\frac{\varepsilon b^2}{4|a|}\right) = \varepsilon\end{aligned}$$

See the text for the proofs of the other 3 properties.

Note: There is a mistake (call it an omission) in the above proof. Can you find it?

Basic Examples

Theorem

(a) $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for $p > 0$.

(b) $\lim_{n\to\infty} a^n = 0$ if $|a| < 1$.

(c) $\lim_{n\to\infty} n^{1/n} = 1$.

(d) $\lim_{n\to\infty} a^{1/n} = 1$ for $a > 0$.

We prove (c) below. See the text for the remaining proofs.

Basic Examples (cont)

To prove (c), we let $a_n = n^{1/n} - 1$ and notice that $a_n > 0$ for $n > 1$. Rearranging we obtain

$$
\begin{aligned}
n^{1/n} &= 1 + a_n \\
\implies n &= (1 + a_n)^n \\
&= 1 + na_n + \frac{n(n-1)}{2}\, a_n^2 + \text{positive terms} \\
&> 1 + \frac{n(n-1)}{2}\, a_n^2
\end{aligned}
$$

It follows that $0 < a_n^2 < 2/n$ so that $a_n^2 \to 0$ as $n \to \infty$ by the Squeeze Law (see exercise 8.5). Notice that by the above example, that $a_n \to 0$. Thus

$$
\begin{aligned}
\lim_{n\to\infty} n^{1/n} &= \lim_{n\to\infty} (n^{1/n} - 1 + 1) \\
&= \lim_{n\to\infty} a_n + \lim_{n\to\infty} 1 \\
&= 0 + 1
\end{aligned}
$$

Infinite Limits

Definition We write $\lim_{n\to\infty} a_n = \infty$ provided that for each $M > 0$ there exists an N such that $n > N$ implies $a_n > M$.

Roughly speaking, the above definition suggests that the terms in the sequence eventually exceed any upper bound. Such limits are said to *diverge to infinity*. **Note:** There is a similar definition for diverging to negative infinity. See the text.

Here is a useful characterization.

Theorem Let a_n be a sequence of positive numbers. Then $\lim_{n\to\infty} a_n = \infty$ if and only if $\lim_{n\to\infty} 1/a_n = 0$.

See the text for a proof.

Example $\lim_{n\to\infty} a^n = \infty$ for $a > 1$.

Observe that if $a > 1$ then $1/a < 1$ and we could prove this by appealing to the last theorem and Part b from the example above. However, with Bernoulli's inequality, the direct proof is almost trivial.

Write $a = 1 + c$ where $c > 0$. Then by Bernoulli's Inequality we have

$$a^n = (1 + c)^n > 1 + nc$$

Now let $M > 1$. By the Archimedian Property, there is a natural number N such that $Nc > M - 1$. It follows that for all $n > N$

$$\begin{aligned} a^n &= (1 + c)^n > 1 + nc \\ &> 1 + Nc \\ &> M \end{aligned}$$

as desired.

Notice that together with the **useful characterization** above, this last result now establishes Part b from the basic examples theorem.