The Exponential Function

In this section we will define the **Exponential** function by the rule

$$\exp(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n \tag{1}$$

Along the way, prove a collection of intermediate results, many of which are important in their own right.

Proposition 1. There exists a real number, $2 < e < 4$ such that

$$\left(1 + \frac{1}{n}\right)^n \nearrow e \quad \text{as} \quad n \to \infty$$

Remark. The notation $b_n \nearrow b$ as $n \to \infty$ is shorthand for $b_n \leq b_{n+1}$ and $\lim_{n\to\infty} b_n = b$.

The limit e, called **Euler's Constant**, can be approximated to a high degree of accuracy. For example,

$$e \approx 2.718281828459045235360287471352662497757247093699959$$

to 50 decimal places.

Before we prove Proposition 1, we need a few intermediate results. If $a > -1$ then

$$(1 + a)^n \geq 1 + na, \tag{2}$$

for $n \in \mathbb{N}$. This is known as **Bernoulli's Inequality**. We will prove this by induction on n. For $n = 1$ we actually have equality. Now suppose that (2) holds for $n = k$. Then

$$\begin{aligned}(1+a)^{k+1} &= (1+a)^k(1+a)\\ &\geq (1+ka)(1+a), && \text{(since } 1 + a > 0)\\ &= 1 + ka + a + ka^2\\ &\geq 1 + (k+1)a\end{aligned}$$

Here the last inequality follows since $ka^2 \geq 0$ and (2) is established.

Lemma 2. For $n \in \mathbb{N}$ we have

(i) $(1 + 1/n)^n$ is increasing.

(ii) $(1 + 1/n)^{n+1}$ is decreasing.

(iii) $2 \leq (1 + 1/n)^n < (1 + 1/n)^{n+1} \leq 4$

Proof. To prove (i) we let $b_n = (1 + 1/n)^n$. We need to show that $b_n < b_{n+1}$. Thus

$$\begin{aligned}\frac{b_{n+1}}{b_n} &= \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^n}\\ &= \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^{n+1}} \left(1 + \frac{1}{n}\right)\\ &= \left(\frac{n^2 + 2n}{n^2 + 2n + 1}\right)^{n+1} \left(1 + \frac{1}{n}\right)\\ &= \left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \left(1 + \frac{1}{n}\right)\\ &\geq \left(1 - \frac{1}{n+1}\right)\left(1 + \frac{1}{n}\right), && \text{(by (2))}\\ &= 1 - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n(n+1)}\\ &= 1\end{aligned}$$

The proof of (ii) is similar. The middle inequality in (iii) is obvious since $(1 + n^{-1}) > 1$. Also, direct calculation and (i) shows that

$$2 = \left(1 + \frac{1}{1}\right)^1 = b_1 < b_n, \quad \text{for all } n \in \mathbb{N}$$

The right-hand inequality is obtained in a similar fashion. □

Proof (of Proposition 1). This follows immediately from Lemma 2 and the Monotone Convergence Theorem. □

Note: From Proposition 1 we see that

$$\left(1 + \frac{1}{n}\right)^n < e, \qquad \text{for all } n \in \mathbb{N} \tag{3}$$

Lemma 3. Let $n \in \mathbb{N}$ and $j \in \mathbb{Z}$ with $0 \leq j \leq n$. Then

$$\binom{n+1}{j}\frac{1}{(n+1)^j} \geq \binom{n}{j}\frac{1}{n^j} \tag{4}$$

Proof. Let b_n^j denote the right-hand side of (4). Then $b_n^0 = b_n^1 = 1$ for all $n \in \mathbb{N}$. Now for $1 < j \leq n$, a routine calculation yields

$$b_{n+1}^j - b_n^j = \frac{(n+1)!}{j!(n+1-j)!(n+1)^j n^j}\left[n^j - (n+1)^{j-1}(n+1-j)\right]$$

So it's enough to show the quantity in brackets is not less than 0. Now

$$\begin{aligned} n^j - (n+1)^{j-1}(n+1-j) &= n^j - (n+1)^j + j(n+1)^{j-1} \\ &= j(n+1)^{j-1} - \left\{n^{j-1} + n^{j-2}(n+1) + \cdots + (n+1)^{j-1}\right\} \\ &= \left\{(n+1)^{j-1} - n^{j-1}\right\} + \left\{(n+1)^{j-1} - n^{j-2}(n+1)\right\} + \cdots \\ &\qquad \cdots + \left\{(n+1)^{j-1} - (n+1)^{j-1}\right\} \\ &\geq 0 \end{aligned}$$

since each of the braced quantities is nonnegative. This proves the lemma. □

Proposition 4. A monotone sequence $\{b_n\}$ converges if and only if it contains a convergent subsequence.

Proof. The only if part is clear. Now suppose that $\{b_n\}$ is an increasing sequence with a convergent subsequence, say $\{b_{n_k}\}$ and let $M > 0$. If $\{b_n\}$ is not bounded above, then there is an $N \in \mathbb{N}$ such that $b_N > M$. It follows that for all $n \geq N$, $b_n \geq b_N > M$. Hence $\{b_{n_k}\}$ is not bounded above. This is impossible. The result now follows by the Monotone Convergence Theorem. □

Lemma 5. Let $x \geq 0$. Then for each $n \in \mathbb{N}$

$$\left(1 + \frac{x}{n}\right)^n \leq \left(1 + \frac{x}{n+1}\right)^{n+1} \tag{5}$$

Proof. We clearly have equality when $x = 0$. Now suppose that $x > 0$ and let

$$a_n(x) = \left(1 + \frac{x}{n}\right)^n$$

From the Binomial Theorem and borrowing the notation from Lemma 3 we have

$$a_n(x) = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{x}{n}\right)^j = \sum_{j=0}^{n} b_n^j\, x^j$$

Then

$$\begin{aligned} a_{n+1}(x) - a_n(x) &= \sum_{j=0}^{n+1} b_{n+1}^j\, x^j - \sum_{j=0}^{n} b_n^j\, x^j \\ &= \sum_{j=0}^{n} \left(b_{n+1}^j - b_n^j\right) x^j + \left(\frac{x}{n+1}\right)^{n+1} \\ &\geq \sum_{j=0}^{n} \left(b_{n+1}^j - b_n^j\right) x^j \\ &\geq 0 \end{aligned}$$

Here the last two lines follow from Lemma 3 and the fact $x^j > 0$. This establishes (5). □

Lemma 6. Let $p, q \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \left(1 + \frac{p/q}{n}\right)^n = e^{p/q} \tag{6}$$

$$\lim_{n\to\infty} \left(1 - \frac{p/q}{n}\right)^n = e^{-p/q} \tag{7}$$

Proof. Let $p, q \in \mathbb{N}$ and define

$$a_n(x) = \left(1 + \frac{x}{n}\right)^n$$

Also, let $a_n = a_n(1)$ and $k \in \mathbb{N}$. Then

$$a_{kq} = \left(1 + \frac{1}{kq}\right)^{kq} = \left(1 + \frac{p/q}{kp}\right)^{kq}$$

So by Proposition 1,

$$a_{kq} \to e \quad \text{as} \quad k \to \infty.$$

It follows that

$$\begin{aligned} \lim_{k\to\infty} a_{kp}(p/q) &= \lim_{k\to\infty} \left(1 + \frac{p/q}{kp}\right)^{kp} \\ &= \lim_{k\to\infty} \left\{\left(1 + \frac{1}{kq}\right)^{kq}\right\}^{p/q} \\ &= \lim_{k\to\infty} (a_{n_k})^{p/q} \\ &= e^{p/q} \end{aligned}$$

Thus $a_{kp}(p/q)$ is a convergent subsequence of the increasing sequence $a_n(p/q)$. Hence (6) now follows by Proposition 4.

The limit in (7) is an easy consequence of the next theorem. □

Remark. As we saw above,

$$\left(1 + \frac{p/q}{n}\right)^n < e^{p/q}$$

for all $n \in \mathbb{N}$.

Theorem 7. Suppose that $b_n \geq 0$ for each $n \in \mathbb{N}$ and that $\lim_{n\to\infty} nb_n = 0$. Then

(a) $\lim\limits_{n\to\infty} (1 + b_n)^n = 1$, and

(b) $\lim\limits_{n\to\infty} (1 - b_n)^n = 1$.

In addition, suppose that $\lim_{n\to\infty} a_n = 0$ and that $\lim_{n\to\infty}(1 + a_n)^n$ is finite. Then

(c) $\lim_{n\to\infty}(1+a_n+b_n)^n = \lim_{n\to\infty}(1+a_n)^n$

Proof. Let $1 > \varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $nb_n = |nb_n| < \varepsilon/2$. Using the Binomial Theorem we see that

$$\begin{aligned} 1 \leq (1+b_n)^n &= 1 + \binom{n}{1} b_n + \binom{n}{2} b_n^2 + \cdots + b_n^n \\ &= 1 + nb_n + \frac{n(n-1)}{2} b_n^2 + \cdots + b_n^n \\ &= 1+ \end{aligned}$$

Hence $n \geq N$ implies

$$\begin{aligned} (1+b_n)^n &< 1 + \frac{n}{n}\frac{\varepsilon}{2} + \frac{n(n-1)}{2n^2}\frac{\varepsilon^2}{2^2} + \cdots + \frac{1}{n^n}\frac{\varepsilon^n}{2^n} \\ &< 1 + \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \cdots + \frac{\varepsilon}{2^n} \\ &= 1 + \varepsilon \sum_{j=1}^{n} \frac{1}{2^j} \\ &< 1 + \varepsilon \end{aligned}$$

In other words, for all $n \geq N$

$$|(1+b_n)^n - 1| < \varepsilon$$

and part (a) is established.

To prove (b), let $c_n = b_n/(1-b_n)$. Then by the limit laws

$$\lim_{n\to\infty} nc_n = \lim_{n\to\infty} \frac{nb_n}{1-b_n} = \frac{\lim_{n\to\infty} nb_n}{1-\lim_{n\to\infty} b_n} = \frac{0}{1-0}$$

Now by (a) we have

$$\begin{aligned} \lim_{n\to\infty}(1-b_n)^{-n} &= \lim_{n\to\infty}\left(\frac{1}{1-b_n}\right)^n \\ &= \lim_{n\to\infty}\left(\frac{1-b_n}{1-b_n} + \frac{b_n}{1-b_n}\right)^n \\ &= \lim_{n\to\infty}(1+c_n)^n = 1 \end{aligned}$$

Once again, by the limit laws

$$\lim_{n\to\infty}(1-b_n)^n = \left(\lim_{n\to\infty}(1-b_n)^{-n}\right)^{-1} = 1$$

To prove part (c), notice that $1 + a_n \neq 0$ for n sufficiently large and hence

$$\lim_{n\to\infty} n\frac{b_n}{1+a_n} = \lim_{n\to\infty} \frac{nb_n}{1+a_n} = 0$$

So by part (a) and the limit laws

$$\begin{aligned} \lim_{n\to\infty}(1+a_n+b_n)^n &= \lim_{n\to\infty}(1+a_n)^n\left(1+\frac{b_n}{1+a_n}\right)^n \\ &= \lim_{n\to\infty}(1+a_n)^n \lim_{n\to\infty}\left(1+\frac{b_n}{1+a_n}\right)^n \\ &= \lim_{n\to\infty}(1+a_n)^n \end{aligned}$$

□

Now to prove (7), let $b_n = \dfrac{(p/q)^2}{n^2}$. Then $\lim_{n\to\infty} nb_n = 0$ and hence,

$$\begin{aligned} \lim_{n\to\infty} a_n(-p/q) &= \lim_{n\to\infty} a_n(-p/q)\frac{a_n(p/q)}{a_n(p/q)} \\ &= \lim_{n\to\infty}\left(1-\frac{(p/q)^2}{n^2}\right)^n \frac{1}{a_n(p/q)} \\ &= \lim_{n\to\infty}(1-b_n)^n \lim_{n\to\infty}\frac{1}{a_n(p/q)} \\ &= \frac{1}{e^{p/q}} \end{aligned}$$

Here we have applied Theorem 7, the limit laws, and (6).

Theorem 8. The exponential function

$$\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \tag{8}$$

is a well-defined real number for each $x \in \mathbb{R}$. Moreover, for $x, y \in \mathbb{R}$ we have

(a) $\exp(x) > 0$. In particular, $x > 0$ implies $\exp(x) > 1$.

(b) $\exp(x)\exp(-x) = 1$

(c) $\exp(x)\exp(y) = \exp(x+y)$

(d) $x < y$ implies $\exp(x) < \exp(y)$

Note: We have already proven (8) for $x \in \mathbb{Q}$.

Proof. Now let $x > 0$. Then by the Archimedean Property, there exists an $N \in \mathbb{N}$ such that $N > x$. Now for each $n \in \mathbb{N}$

$$a_n(x) =^{\text{def}} \left(1+\frac{x}{n}\right)^n < \left(1+\frac{N}{n}\right)^n < e^N < \infty$$

Now by Lemma 5 $a_n(x)$ is an increasing sequence. Hence, by the Monotone Convergence Theorem,

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \exp(x) \leq e^N$$

Also, for all $n \in \mathbb{N}$

$$1 + n\,\frac{x}{n} \leq \left(1+\frac{x}{n}\right)^n \tag{9}$$

by Bernoulli's Inequality. Thus

$$1 < 1 + x \leq \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \exp(x)$$

since x is positive. Now by Theorem 7

$$\begin{aligned}\exp(-x) &= \lim_{n\to\infty}\left(1+\frac{-x}{n}\right)^n = \lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n \frac{(1+\frac{x}{n})^n}{(1+\frac{x}{n})^n}\\ &= \lim_{n\to\infty}\left(1-\frac{x^2}{n^2}\right)^n \frac{1}{(1+\frac{x}{n})^n}\\ &= \lim_{n\to\infty}\left(1-\frac{x^2}{n^2}\right)^n \lim_{n\to\infty}\frac{1}{(1+\frac{x}{n})^n}\\ &= 1\cdot\frac{1}{\exp(x)}\end{aligned}$$

This establishes (8) and parts (a) and (b). To prove (c), let $x, y \in \mathbb{R}$. Then $\lim_{n\to\infty} n(xy/n^2) = 0$ and by Theorem 7(c) we have

$$\begin{aligned}\exp(x)\exp(y) &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \lim_{n\to\infty}\left(1+\frac{y}{n}\right)^n\\ &= \lim_{n\to\infty}\left(1+\frac{x+y}{n}+\frac{xy}{n^2}\right)^n\\ &= \lim_{n\to\infty}\left(1+\frac{x+y}{n}\right)^n\\ &= \exp(x+y)\end{aligned}$$

To prove (d), let $x < y$. Then $y - x > 0$ and by parts (c) and (a)

$$\exp(y) - \exp(x) = \exp(x)(\exp(y-x) - 1) > 0$$

□

Motivated by this (and the results from Lemma 6), we make the following definition.

Definition. Let $x \in \mathbb{R}$ and let e represent Euler's constant. We define e^x by

$$e^x = \exp(x) \tag{10}$$

Properties of the Exponential Function

We first catalogue a few important inequalities.

Lemma 9.

(a) $1 + x \leq e^x$ for all $x \in \mathbb{R}$, and

(b) $e^x \leq \dfrac{1}{1-x}$ for $x < 1$.

Proof. We have equality in both when $x = 0$.

The inequality in part (a) is obvious if $x \leq -1$ since the left-hand side is nonpositive. If $x > -1$ then by (2)

$$\left(1+\frac{x}{n}\right)^n \geq 1 + n\,\frac{x}{n} = 1 + x$$

for all $n \in \mathbb{N}$. Thus

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \geq 1 + x$$

To prove part (b), suppose that $x < 1$. Then $1 - x > 0$ and by part (a)

$$e^{-x} \geq 1 - x > 0$$

Rearranging, we obtain (b). □

The following theorem is an immediate consequence of Lemma 9.

Theorem 10.

(a) $\displaystyle\lim_{x\to 0} e^x = 1$

(b) $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(c) $\displaystyle\lim_{x\to\infty} e^x = \infty$ and $\displaystyle\lim_{x\to-\infty} e^x = 0$.

Remark. Since $e^0 = 1$, the limit in (a) says that the exponential function is continuous at the origin.

Proof. To prove part (a), observe that for all $x \in (-1/2, 1/2)$ we have

$$1 + x \leq e^x \leq \frac{1}{1-x} \tag{11}$$

by Lemma 9. Now let $x \to 0$ and invoke the Squeeze Law.

To prove (b), notice that (11) implies

$$x \le e^x - 1 \le \frac{1}{1-x} - 1 = \frac{x}{1-x}$$

Dividing by positive x yields

$$1 \le \frac{e^x - 1}{x} \le \frac{1}{1-x}$$

On the other hand, if $x < 0$ then we obtain the reverse inequality

$$1 \ge \frac{e^x - 1}{x} \ge \frac{1}{1-x}$$

Now let $x \to 0^+$ and $x \to 0^-$ respectively in the above inequalities. Part (b) now follows by the Squeeze Law.

Part (c) is an immediate consequence of Lemma 9. For example, let $M > 0$. Then $\exp(M) \ge M + 1 > M$. The proof of the second limit is nearly as trivial. □

The next 2 theorems make clear the importance of Theorem 10.

Theorem 11. The exponential function $\exp(x)$ is a continuous, strictly increasing function from $\mathbb{R}$ onto $(0, \infty)$.

Proof. We have already seen that the exponential function is strictly increasing (see Theorem 8). Now let $x \in \mathbb{R}$. Then by Theorem 10

$$\lim_{h\to 0} e^{x+h} = \lim_{h\to 0} e^x e^h = e^x \lim_{h\to 0} e^h = e^x$$

In other words, the exponential function is continuous.

Finally, let $L > 0$. Then by Theorem 10(c), there exist real numbers a and b such that $e^a < L < e^b$. So by the Intermediate Value Theorem there is a $c \in (a, b)$ such that $e^c = L$. □

Theorem 12. The exponential function $\exp(x)$ is differentiable. In fact,

$$\frac{de^x}{dx} = e^x$$

Proof. Let $x \in \mathbb{R}$. Once again, by Theorem 10 we have

$$\frac{de^x}{dx} = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x e^h - e^x}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x$$

□

Now let $x \in \mathbb{R}$. We are now able to define a^x for arbitrary positive numbers a. Of course, $1^x = 1$.

Definition. Now let $a > 0$, $a \ne 1$. By Theorem 11, there exists a real number c such that $e^c = a$. For each $x \in \mathbb{R}$ we define

(12) $$a^x = e^{xc} = \lim_{n\to\infty} \left(1 + \frac{xc}{n}\right)^n$$

Note: c is called the (natural) logarithm of a and is denoted $c = \ln a$.

Remark. It turns out that $f(x) = a^x$ is a differentiable function from $\mathbb{R}$ onto $(0, \infty)$, and if $a = e^c$ then

$$\frac{da^x}{dx} = ca^x$$

Also, f is strictly increasing when $a > 1$. Otherwise, f is strictly decreasing.