Series Tests for Convergence - Summary

Recall

Definition. Given the **infinite series**

(1) $$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

we define the following. The number a_n is called the **nth term** of the series. It is also called the **summand**. The **nth partial sum** of the series is denoted by s_n and is defined by

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

Notice that the partial sums generate a new sequence, the so-called **sequence of partial sums**, $\{s_n\}$. Now if this new sequence converges to a limit, say $L \in \mathbb{R}$, we say that the series (1) converges and that its **sum** is L. Specifically,

(2) $$s_n \to L \text{ as } n \to \infty \quad \Longrightarrow \quad \sum_{n=1}^{\infty} a_n = L$$

In other words,

(3) $$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} s_n$$

whenever the limit exists. Otherwise, the series **diverges**.

We have the following general test for convergence.

Theorem 1. Cauchy Criterion for Series. The series $\sum_{n=1}^{\infty} a_n$ converges if and only if, for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > m \geq N$ we have

$$|a_{m+1} + a_{m+2} + \cdots a_n| = \left| \sum_{j=m+1}^{n} a_j \right| < \varepsilon$$

Proof. Notice that

$$s_n - s_m = \sum_{j=1}^{n} a_j - \sum_{j=1}^{m} a_j = a_{m+1} + a_{m+2} + \cdots a_n$$

Now apply the Cauchy Criterion for sequences to $\{s_n\}$. □

We summarize the various convergence *tests* for infinite series. Suppose that $a_n \geq 0$ for all $n \geq N, (N \in \mathbb{Z})$. To test the series $\sum a_n$ for convergence (or divergence) we have the following.

1. **n-Term Test (for Divergence).**

   If $a_n \nrightarrow 0$ then $\sum_n a_n$ diverges.

   Remark. This test is valid for any series, not just series with nonnegative terms.

2. **Cauchy Condensation Test.** If $\{a_n\}$ is a nonincreasing sequence that converges to 0. Then

   $$\sum_n a_n < \infty \text{ iff } \sum_n 2^n a_{2^n} < \infty$$

3. **Comparison Test.**
   (a) $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n \geq N$ for some positive integer N.
   (b) $\sum a_n$ diverges if there is a divergent series $\sum d_n$ with $a_n \geq d_n \geq 0$ for all $n \geq N$ for some positive integer N.

4. **Limit Comparison Test**. Let $a_n > 0$ and $b_n > 0$ for all $n \geq N$.
   (a) Suppose that $\dfrac{a_n}{b_n} \to \delta \in [0, \infty)$. If $\sum b_n$ converges then so does $\sum a_n$.
   (b) Suppose that $\dfrac{a_n}{b_n} \to \delta \in (0, \infty]$. If $\sum b_n$ diverges then so does $\sum a_n$.

5. **Ratio Test.** Let $\sum a_n$ be a series of positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

   Then
   (a) the series *converges* if $\rho < 1$,
   (b) the series *diverges* if $\rho > 1$ or ρ is infinite,
   (c) the test is *inconclusive* if $\rho = 1$.

6. **Root Test.** Suppose that $a_n \geq 0$ for $n \geq N$ and

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$$

   Then
   (a) the series *converges* if $\rho < 1$,
   (b) the series *diverges* if $\rho > 1$ or ρ is infinite,
   (c) the test is *inconclusive* if $\rho = 1$.

7. **Alternating Series Test (Leibnitz's Theorem).** Let N be a positive integer. The alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

   converges provided that the following three conditions are satisfied.
   (a) $a_n > 0$ for all $n \geq N$.
   (b) $a_n \geq a_{n+1}$ for all $n \geq N$.
   (c) $a_n \to 0$.

Example 1. Does the series below converge or diverge. *Give reasons for your answer.*

$$\sum_{n=2}^{\infty} \frac{1}{1 + (\ln n)^3}$$

We claim that the series diverges by the Cauchy Condensation Test. Let

$$a_n = \frac{1}{1 + (\ln n)^3}$$

Notice that $a_n \to 0$ and, since the denominator is increasing, we clearly have $a_n \geq a_{n+1}$ so that the CCT applies. So the series $\sum a_n$ and the series $\sum 2^n a_{2^n}$ converge or diverge together. Now

$$\sum_{n=2}^{\infty} 2^n a_{2^n} = \sum_{n=2}^{\infty} \frac{2^n}{1 + (\ln 2^n)^3} = \sum_{n=2}^{\infty} \frac{2^n}{1 + (\ln 2)^3 \; n^3}$$

but

$$\lim_{n\to\infty} \frac{2^n}{1 + (\ln 2)^3 \; n^3} = \infty$$

So the series $\sum 2^n a_{2^n}$ diverges by the nth-term test. The result follows.

Example 2. Do the following series converge or diverge. Justify your claim.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+9}}$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n2^n}$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^{n-1}+2}$

d. $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$

e. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{(2n)!}$

f. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n!)^2}{(2n)!}$

Example 3. Do the following series converge or diverge. Justify your claim.

a. $\displaystyle\sum_{n=1}^{\infty} n\, e^{-n^2}$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+2}$

c. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$

d. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(1/n)}{n^2}$

e. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n\, n!}{(2n)!}$

f. $\displaystyle\sum_{n=1}^{\infty} (1-\cos(1/n))$