16.4 Green's Theorem (cont)

Divergence

Definition. Divergence (Flux Density)

If $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is a vector field in $\mathbb{R}^3$ and if the partial derivatives of M, N, and P exist, then the **divergence** of $\mathbf{F}$ is the scalar

$$\begin{aligned} \text{div}\,\mathbf{F} &= \nabla \cdot \mathbf{F} \\ &= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \end{aligned}$$

Notice that the divergence is **real**-valued.

Example 1.

Find the divergence of $\mathbf{F} = x^2y\,\mathbf{i} + 2xy\,\mathbf{j} + z^3\,\mathbf{k}$.

$$\begin{aligned} \text{div}\,\mathbf{F} &= \nabla \cdot \mathbf{F} \\ &= \frac{\partial\left(x^2y\right)}{\partial x} + \frac{\partial\,(2xy)}{\partial y} + \frac{\partial\left(z^3\right)}{\partial z} \\ &= 2xy + 2x + 3z^2 \end{aligned}$$

Now suppose that $\mathbf{F}$ is a velocity field of a fluid flow. Then, for example,

$$\text{div}\,\mathbf{F}(1,\,2,\,1) = 2(1)(2) + 2(1) + 3(1)^2 = 9$$

implies that fluid is being piped away from the point $(1,2,1)$.

Flux Across a Plane Curve

Definition. If C is a smooth **closed** curve in the domain of a continuous vector field $\mathbf{F} = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$ in the plane and if $\mathbf{n}$ is the outward-pointing normal vector on C, then the **flux** of $\mathbf{F}$ across C is

$$\text{Flux} = \oint_C \mathbf{F}\cdot\mathbf{n}\,ds$$

Notice that the flux of $\mathbf{F}$ across C is the line integral of the *scalar component* of $\mathbf{F}$ in the direction of outward normal.

Now suppose that C is parameterized by

$$x = x(t), \quad y = y(t), \qquad a \le t \le b$$

traces the curve in the counterclockwise direction exactly once.

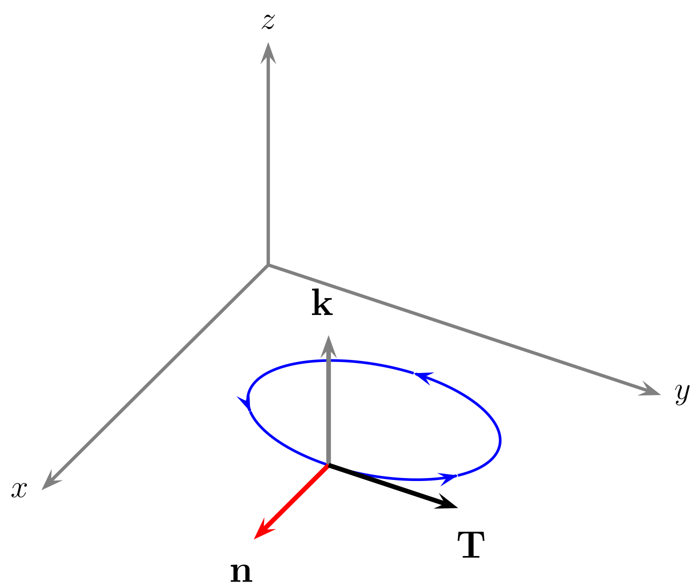


Figure 1: Relationship between $\mathbf{T}, \mathbf{n}, \mathbf{k}$

In chapter 13 we saw that the unit tangent vector, $\mathbf{T}$ was given by

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}$$

Notice that $\mathbf{n} = \mathbf{T} \times \mathbf{k}$. See Figure 1. Thus

$$\begin{aligned}\mathbf{n} &= \mathbf{T} \times \mathbf{k} \\ &= \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}\right) \times \mathbf{k} \\ &= \frac{dx}{ds}(-\mathbf{j}) + \frac{dy}{ds}\mathbf{i}\end{aligned}$$

or

$$= \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}$$

If

$$\mathbf{F} = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j},$$

then

$$\mathbf{F}\cdot\mathbf{n} = M(x,\ y)\,\frac{dy}{ds} - N(x,\ y)\,\frac{dx}{ds}$$

It follows that

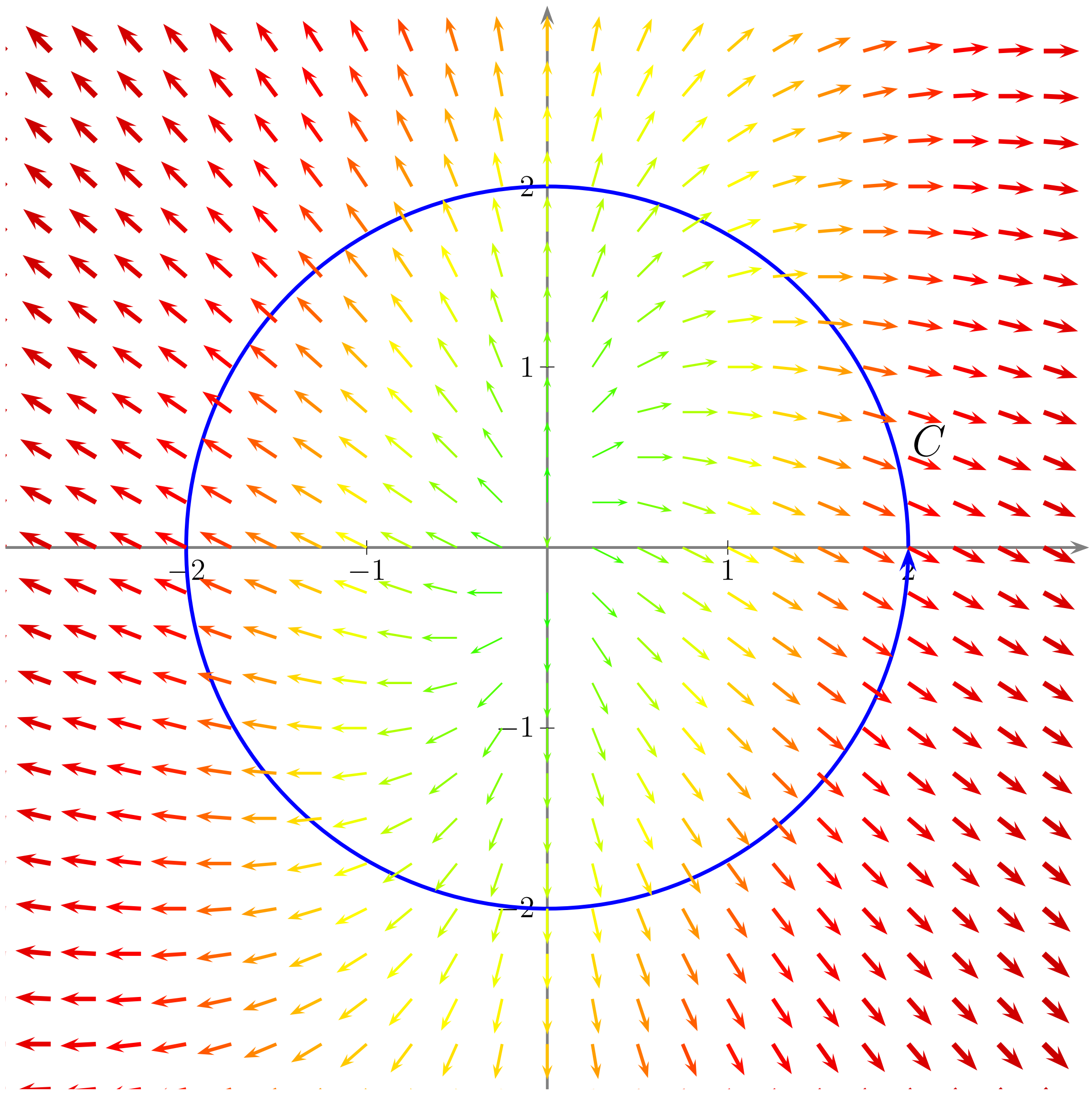
$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C \left(M\,\frac{dy}{ds} - N\,\frac{dx}{ds}\right)\,ds$$

(1) $$= \oint_C M\,dy - N\,dx$$

Example 2. Computing Flux

Let $\mathbf{F} = 2x\,\mathbf{i} + (y - x)\,\mathbf{j}$. Find the outward flux of the field $\mathbf{F}$ across the circle.

$$C\colon\ \mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}, \quad 0 \le t \le 2\pi$$



$$
\begin{aligned}
x &= a\cos t, \quad dx = -a\sin t\,dt \\
y &= a\sin t, \quad dy = a\cos t\,dt \\
M &= 2x = 2a\cos t \\
N &= y - x = a\sin t - a\cos t
\end{aligned}
$$

Thus

$$
\begin{aligned}
\oint_C \mathbf{F}\cdot\mathbf{n}\,ds &= \oint_C M\,dy - N\,dx \\
&= \int_0^{2\pi} 2a\cos t\,a\cos t\,dt + (a\sin t - a\cos t)\,a\sin t\,dt \\
&= a^2\int_0^{2\pi} \left(2\cos^2 t + \sin^2 t - \sin t\ \cos t\right) dt \\
&= a^2\int_0^{2\pi} \left(1 + \cos^2 t - \sin t\ \cos t\right) dt \\
&= a^2\int_0^{2\pi} \left(1 - \sin t\ \cos t + \frac{1}{2}\left(1 + \cos 2t\right)\right) dt \\
&= a^2\int_0^{2\pi} \left(\frac{3}{2} - \sin t\ \cos t + \frac{\cos 2t}{2}\right) dt \\
&= a^2\left(\frac{3t}{2} - \frac{\sin^2 t}{2} + \frac{\sin 2t}{4}\right)\ \Bigg|_0^{2\pi} = 3a^2\pi
\end{aligned}
$$

Theorem 1. Green's Theorem (Normal Form)

The outward flux of a field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ across a simple closed curve C is equal to the double integral of the *flux density* over the region R enclosed by C. Suppose also that M and N have continuous partial derivatives on an open region that contains R.

(2)
$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C M\,dy - N\,dx$$

(3)
$$= \iint_R \underbrace{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)}_{\text{divergence (flux density)}} \,dx\,dy$$

or, more conveniently,

$$= \iint_R \underbrace{\nabla\cdot\mathbf{F}}_{\text{divergence}} \,dx\,dy$$

Proof. This one is easy. By the tangential form of Green's Theorem, we have

$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathbf{n}\,ds &= \oint_C M\,dy - N\,dx \\ &= \oint_C -N\,dx + M\,dy \\ &= \iint_R \left(\frac{\partial M}{\partial x} - \frac{\partial(-N)}{\partial y}\right) dx\,dy \\ &= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy\end{aligned}$$

□

Example 3. Consider the velocity vector field below.

$$\mathbf{F} = 5y\,\mathbf{j}$$

over the unit square R as shown in Figure 2. Once again, we imagine the field represents a thin fluid flowing over the xy-plane and the units of $\mathbf{F}$ are expressed in ft/sec.

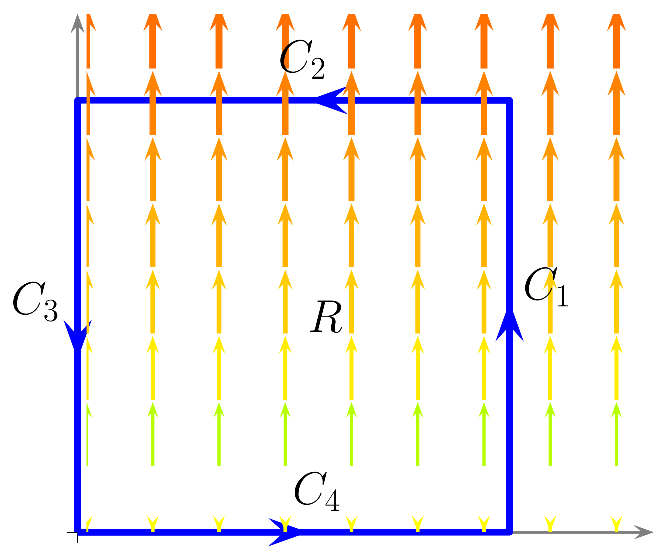


Figure 2: Velocity Field $5y\,\mathbf{j}$

What can you say about the flux density (see Figure 2) at each point within the region R?

Now find the outward flux for the velocity field $\mathbf{F}$ across the region R in two different ways.

We first proceed directly, that is, we evaluate the line integral $\int_{\partial R} \mathbf{F}\cdot\mathbf{n}\,ds$.

$$\oint_{\partial R} \mathbf{F}\cdot\mathbf{n}\,ds = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} \mathbf{F}\cdot\mathbf{n}\,ds$$

It is pretty easy to see that the flux across each of the line segments C_1, C_3, C_4 is zero. Why? And we leave it as an easy exercise to show that

$$\oint_{\partial R} \mathbf{F} \cdot \mathbf{n}\, ds = \int_{C_2} \mathbf{F} \cdot \mathbf{n}\, ds = 5$$

Now calculate the flux using (the normal form) of Green's Theorem. We have

$$\begin{aligned}
\oint_{\partial R} \mathbf{F} \cdot \mathbf{n}\, ds &= \iint_R \left(\frac{\partial(0)}{\partial x} + \frac{\partial(5y)}{\partial y} \right) dx\, dy \\
&= 5 \iint_R dx\, dy \\
&= 5 \times \text{area of } R \\
&= 5
\end{aligned}$$

Is there a physical interpretation of this result?

After examining the units, we see that flux is 5 ft^2/sec. Let's make a further assumption that the thin fluid is water with a depth of 1/2 inch. In that case, the flux calculation tells us that water is being piped out of (or away from) the region R at a rate of 5/24 cubic ft per second or approximately 1.67 gallons per minute.

We will be able to make this example a bit more concrete after we discuss the 3-dimensional analogs of Green's Theorem after the break.

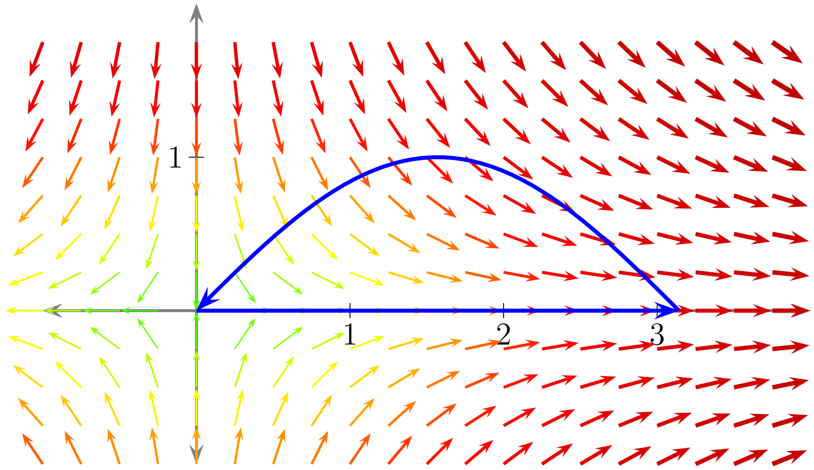


Figure 3: The field $\mathbf{G} = 2x\,\mathbf{i} + (-3y)\,\mathbf{j}$

Example 4. Let $\mathbf{G} = 2x\,\mathbf{i} + (-3y)\,\mathbf{j}$ and evaluate the flux integral $\int_C \mathbf{G}\cdot\mathbf{n}\,ds$. By the normal form of Green's Theorem we have

$$\begin{aligned}\oint_C 2x\,dy - (-3y)\,dx &= \oint_C M\,dy - N\,dx \\ &= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\,dx\,dy \\ &= \int_0^{\pi}\int_0^{\sin x} (2-3)\,dy\,dx \\ &= -\int_0^{\pi} \sin x\,dx \\ &= \cos x\,\Big|_0^{\pi} = -2\end{aligned}$$

Once again, here are both forms of Green's Theorem.

Let C be a piecewise-smooth, simple closed curve in the plane and let R be the region bounded by C (in the plane). Suppose also that M and N have continuous partial derivatives on an open region that contains R. Then

Green's Theorem (Tangential Form)

(4) $$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C M\,dx + N\,dy$$

(5) $$= \iint_R \underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{(\nabla\times\mathbf{F})\cdot\mathbf{k}}\,dx\,dy$$

Green's Theorem (Normal Form)

(6) $$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C M\,dy - N\,dx$$

(7) $$= \iint_R \underbrace{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)}_{\nabla\cdot\mathbf{F}}\,dx\,dy$$

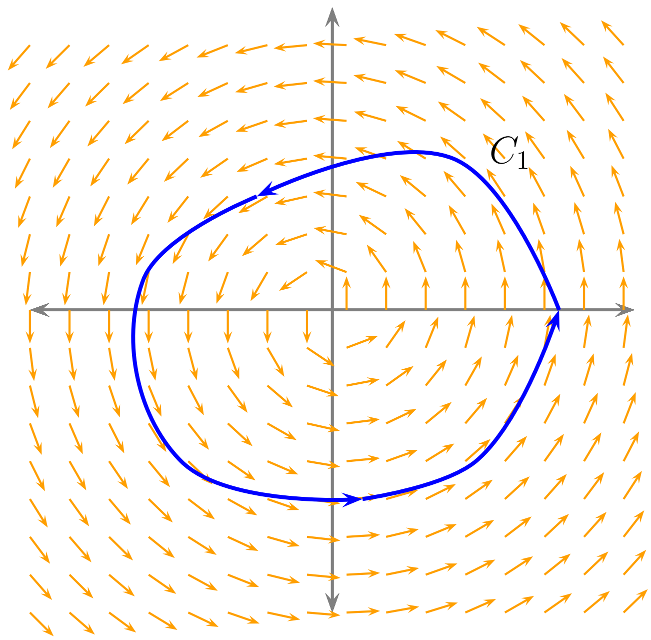


Figure 4: A Spin Field

It turns out that (both forms of) Green's Theorem apply for regions with holes. We illustrate with an example. For a proof, see the text.

Example 5. Let $\mathbf{F} = \frac{-y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j}$ and let C_1 be any positively oriented, piecewise smooth, closed curve that contains the origin. We sketch an example curve in Figure 4.

We claim that

(8) $$\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = 2\pi$$

How do we know this without an explicit description of C_1?

It is easy to see that $\mathbf{F}$ is conservative over any region that does not contain the origin. Now let C_2 be a circle centered at the origin of radius $a > 0$ chosen so that C_2 lies inside of C_1 and let R be the region inside C_1 but outside C_2 (see Figure 5).

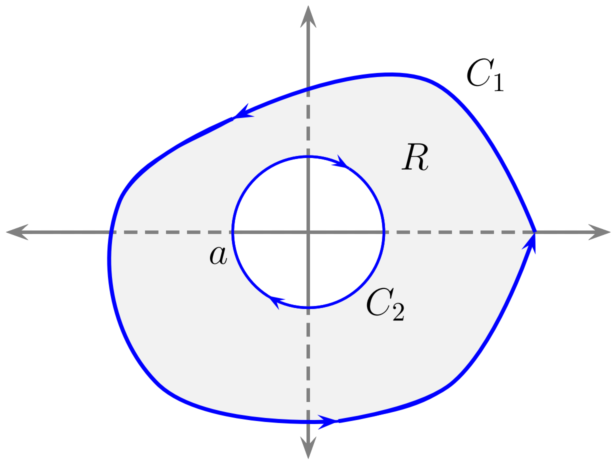


Figure 5: A region with a hole

Notice that we orient C_2 so that R lies to our left we traverse the curve. It is easy to confirm that $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = -2\pi$ (see Example 11 from the previous lecture). Now let $C = C_1 \cup C_2$. If the tangential form of Green's Theorem holds on R (it does), then we have

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy \\ &= 0\end{aligned}$$

Thus

$$\begin{aligned}0 &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\ &= \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} \\ &= \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} - 2\pi\end{aligned}$$

and the result follows.

Example 6. Let T be the triangular region in the first quadrant bounded by the lines $2x + y = 1$, $x = 0$, and $y = 0$. Evaluate the integral below.

$$\oint_{\partial T} y^2\,dx + x^2\,dy$$

We appeal to (the tangential form of) Green's Theorem.

$$\begin{aligned}
\oint_{\partial T} y^2\,dx + x^2\,dy &= \iint_T \left(\frac{\partial(x^2)}{\partial x} - \frac{\partial(y^2)}{\partial y}\right) dy\,dx \\
&= 2\int_0^{1/2}\int_0^{1-2x} x - y\,dy\,dx \\
&= \int_0^{1/2} 6 - 8x^2 - 1\,dx \\
&= -1/12
\end{aligned}$$

We leave it as an exercise to evaluate the line integral directly and also to rewrite the line integral as a flux integral and apply Green's Theorem. Both calculations should yield the same result.

Example 7. Let $a > 0$. Find the outward flux of the velocity field

$$\mathbf{F} = \left(3xy - \frac{x}{1+y^2}\right)\mathbf{i} + (e^x + \tan^{-1} y)\,\mathbf{j}$$

across the upper half of the cardioid **region** R defined by

$$R: \quad r(\theta) \leq a(1+\cos\theta), \quad 0 \leq \theta \leq \pi$$

Let $M = 3xy - \dfrac{x}{1+y^2}$ and $N = e^x + \tan^{-1} y$. Then

$$\begin{aligned}
\oint_{\partial R} \mathbf{F}\cdot\mathbf{n}\,ds &= \oint_{\partial R} M\,dy - N\,dx \\
&= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy \\
&= \iint_R 3y - \frac{1}{1+y^2} + \frac{1}{1+y^2}\,dx\,dy \\
&= \int_0^\pi \int_0^{a(1+\cos\theta)} 3r^2 \sin\theta\,dr\,d\theta \\
&= a^3 \int_0^\pi \sin\theta(1+\cos\theta)^3\,d\theta \\
&= \frac{-a^3}{4}(1+\cos\theta)^4 \Big|_0^\pi \\
&= 4a^3
\end{aligned}$$

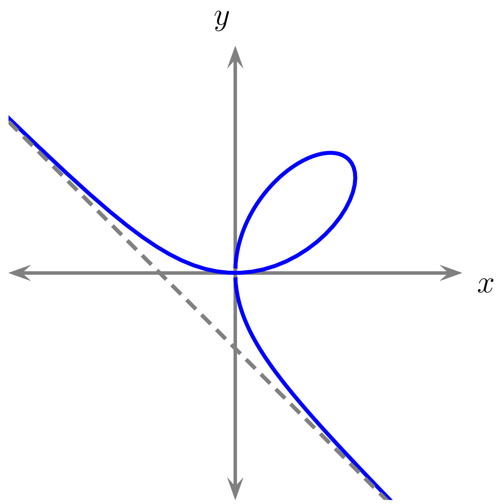


Figure 6: Folium of Descartes (with $a = 1$)

Example 8. Let $a > 0$. In or around 1638, Rene Descartes challenged Pierre de Fermat to find the tangent line at any point along the curve whose parametric equations were given by

(9) $$x = \frac{3at}{1+t^3} \quad \text{and} \quad y = \frac{3at^2}{1+t^3}$$

It is easy to show that the curve can be expressed in rectangular coordinates as

$$x^3 + y^3 = 3axy$$

and it has a slant asymptote $x + y = -a$. These days the problem is often found in a textbook covering first semester calculus, but in 1638 calculus had not yet been discovered.

Use the methods of Example 8 in the previous section to find the area inside the loop C (which happens to lie in quadrant 1). *Hint:* The parametric equations trace out one half of the loop for $0 \leq t \leq 1$.

Referring to (9), notice that $y/x = t$. Now the quotient rule yields

$$dt = \frac{x\,dy - y\,dx}{x^2}$$

or

$$x^2\,dt = x\,dy - y\,dx$$

It follows by (6) (from 16.4p1) that

$$\begin{aligned}\text{area} &= \frac{1}{2}\oint_C x\,dy - y\,dx \\ &= \frac{1}{2}\oint_C x^2\,dt \\ &= \frac{1}{2}\int_0^\infty \left(\frac{3at}{1+t^3}\right)^2\,dt\end{aligned}$$

Following the hint, the last line reduces to

$$\begin{aligned}&= 2 \times \frac{1}{2}\int_0^1 \left(\frac{3at}{1+t^3}\right)^2\,dt \\ &= 3a^2\int_0^1 \frac{3t^2}{(1+t^3)^2}\,dt \\ &= 3a^2\int_1^2 \frac{du}{u^2} \\ &= \frac{3a^2}{2}\end{aligned}$$