16.0 Curl and Divergence

Definition. Circulation Density at a Point in the Plane

The **circulation density** of a vector field $\mathbf{F}(x, y) = M\,\mathbf{i} + N\,\mathbf{j}$ at a point (x, y) is

$$(\text{curl}\,\mathbf{F}) \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \tag{1}$$

It turns out that this notion can be generalized in 3-space. We have the following

Definition. Curl (Circulation Density)

If $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is a vector field in $\mathbb{R}^3$ and if the partial derivatives of M, N, and P exist, then the **curl** of $\mathbf{F}$ is the vector field

$$\begin{aligned}
\text{curl}\,\mathbf{F} &= \nabla \times \mathbf{F} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \\
&= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}
\end{aligned}$$

Notice that curl in space is a **vector**.

There is a related quantity.

Definition. Divergence (Flux Density)

If $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is a vector field in $\mathbb{R}^3$ and if the partial derivatives of M, N, and P exist, then the **divergence** of $\mathbf{F}$ is the scalar

$$\begin{aligned} \text{div}\,\mathbf{F} &= \nabla \cdot \mathbf{F} \\ &= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \end{aligned}$$

Notice that, unlike curl, the divergence of a vector field is **real**-valued. We describe a physical interpretation of divergence in the next example.

Example 1. Find the divergence and curl of the vector field $\mathbf{F} = x^2y\,\mathbf{i} + 2xy\,\mathbf{j} + z^3\,\mathbf{k}$.

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial\left(x^2y\right)}{\partial x} + \frac{\partial\left(2xy\right)}{\partial y} + \frac{\partial\left(z^3\right)}{\partial z}$$

$$= 2xy + 2x + 3z^2$$

Now suppose that $\mathbf{F}$ is a velocity field of a fluid flow. Then, for example,

$$\text{div}\,\mathbf{F}(1,\,2,\,1) = 2(1)(2) + 2(1) + 3(1)^2 = 9$$

implies that fluid is being piped away from the point $(1, 2, 1)$ since the result is positive. To compute the curl we have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & 2xy & z^3 \end{vmatrix}$$

$$= \mathbf{i}\left(\frac{\partial\left(z^3\right)}{\partial y} - \frac{\partial\left(2xy\right)}{\partial z}\right) - \mathbf{j}\left(\frac{\partial\left(z^3\right)}{\partial x} - \frac{\partial\left(x^2y\right)}{\partial z}\right)$$

$$+\ \mathbf{k}\left(\frac{\partial\left(2xy\right)}{\partial x} - \frac{\partial\left(x^2y\right)}{\partial y}\right)$$

$$= \mathbf{i}\,(0-0) - \mathbf{j}\,(0-0) + \mathbf{k}\left(2y - x^2\right)$$

$$= (2y - x^2)\,\mathbf{k}$$

Is $\mathbf{F}$ a conservative vector field?

An Important Identity

$$\text{div}\,\text{curl}\,\mathbf{F} = \mathbf{0}$$

or

$$\nabla \cdot (\nabla \times \mathbf{F}) = \mathbf{0}$$

For a proof, see the text.

Finding Potential Functions

Recall that if $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is conservative, then there is a function f (called the potential function) such that $\nabla f = \mathbf{F}$? To find f we work with the following partial differential equations (PDEs).

$$\frac{\partial f}{\partial x} = M, \quad \frac{\partial f}{\partial y} = N, \quad \frac{\partial f}{\partial z} = P \tag{2}$$

We illustrate below.

Example 2. Show that the vector field below is conservative and find its potential function.

$$\mathbf{F} = \left(\sin y + z e^{xz}\right)\,\mathbf{i} + x\cos y\;\mathbf{j} + x e^{xz}\,\mathbf{k}$$

We first compute the curl.

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin y + z e^{xz} & x\cos y & x e^{xz} \end{vmatrix} \\
&= \mathbf{i}\left(\frac{\partial\left(x e^{xz}\right)}{\partial y} - \frac{\partial\left(x\cos y\right)}{\partial z}\right) - \mathbf{j}\left(\frac{\partial\left(x e^{xz}\right)}{\partial x} - \frac{\partial\left(\sin y + z e^{xz}\right)}{\partial z}\right) \\
&\qquad + \mathbf{k}\left(\frac{\partial\left(x\cos y\right)}{\partial x} - \frac{\partial\left(\sin y + z e^{xz}\right)}{\partial y}\right) \\
&= \mathbf{i}\,(0-0) - \mathbf{j}\left(e^{xz} + xz\,e^{xz} - e^{xz} - xz\,e^{xz}\right) + \mathbf{k}\left(\cos y - \cos y\right) \\
&= \mathbf{0}
\end{aligned}$$

as expected.

To find the potential function, we first can integrate both sides of

$$\frac{\partial f}{\partial x} = \sin y + z e^{xz}$$

to obtain

$$f = x \sin y + e^{xz} + g(y, z)$$

Now comparing the $\mathbf{j}$ component of $\mathbf{F}$ with $\dfrac{\partial f}{\partial y}$ yields

$$x \cos y = x \cos y + \frac{\partial g}{\partial y}$$

It follows that g does not depend on y. In other words

$$f = x \sin y + e^{xz} + h(z)$$

Finally, we compare the $\mathbf{k}$ component of $\mathbf{F}$ with $\dfrac{\partial f}{\partial z}$ to obtain

$$x e^{xz} = x e^{xz} + h'(z)$$

It follows that $h'(z) = 0$ and hence

$$f(x, y, z) = x \sin y + e^{xz} + C$$

where C is an arbitrary constant.

Line Integrals

Recall the two types of line integrals that we encountered in section 16.2.

Let $\mathbf{r}(t)$ be a smooth parametrization of a curve C for $a \le t \le b$ that lies in the domain of a real-valued function f.

Then the **line integral of f over C** is defined by

(3) $$\int_C f(x,y,z)\,ds = \int_a^b f\left(x(t),y(t),z(t)\right)\,|\mathbf{r}'(t)|\,dt$$

Suppose instead that we have a continuous vector field $\mathbf{F}$ defined on a smooth curve C which is parameterized by the vector function $\mathbf{r}(t),\ a \le t \le b$. Then the **line integral of $\mathbf{F}$ along C** is

(4) $$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_{t=a}^{t=b} \mathbf{F}\cdot d\mathbf{r}$$

We must point out the various equivalent ways that (4) can be written.

$$\begin{aligned}
\int_C \mathbf{F}\cdot\mathbf{T}\,ds &= \int_{t=a}^{t=b} \mathbf{F}\cdot\mathbf{T}\,ds \\
&= \int_{t=a}^{t=b} \mathbf{F}\cdot d\mathbf{r} \\
&= \int_a^b \mathbf{F}\cdot \frac{d\mathbf{r}}{dt}\,dt \\
&= \int_a^b \left(M\,\frac{dx}{dt} + N\,\frac{dy}{dt} + P\,\frac{dz}{dt} \right)\, dt \\
&= \int_a^b M\,dx + N\,dy + P\,dz
\end{aligned}$$

Example 3. Evaluate the integrals below. In each case, specify whether the integral is more like (3) or (4).

a. $\int_C xy\,dx + z\,dy + (x+z)dz$

Here C can be parametrized by $\mathbf{r}(t) = \left\langle t^2, t, t^3 \right\rangle,\ 0 \le t \le 1$.

This is more like (4). We have

$$\begin{aligned} xy\,dx &= t^2 \cdot t \cdot 2t\,dt = 2t^4\,dt \\ z\,dy &= t^3\,dt \\ (x+z)\,dz &= (t^2+t^3)\cdot 3t^2\,dt = 3t^4\,dt + 3t^5\,dt \end{aligned}$$

It follows that

$$\begin{aligned} \int_C xy\,dx + z\,dy + (x+z)dz &= \int_0^1 t^3 + 5t^4 + 3t^5\,dt \\ &= \left(\frac{t^4}{4} + t^5 + \frac{t^6}{2}\right)\Bigg|_0^1 \\ &= 7/4 \end{aligned}$$

b. $I = \int_C (\sin y + ze^{xz})\, dx + x\cos y\, dy + xe^{xz}\, dz$

Here C is any smooth curve from $A(2, \pi/2, 0)$ to $B(3, 3\pi/2, 1)$

Notice that if we set

$$f(x, y, z) = x\sin y + e^{xz}$$

then

$$df = (\sin y + ze^{xz})\, dx + x\cos y\, dy + xe^{xz}\, dz \quad \text{(see Ex. 2)}$$

It follows that the integral is path independent. Hence

$$\begin{aligned} I &= \int_A^B df \\ &= f(B) - f(A) \\ &= \left(3\sin\frac{3\pi}{2} + e^3\right) - \left(2\sin\frac{\pi}{2} + 1\right) \\ &= e^3 - 6 \end{aligned}$$

Notice that this integral is more like (4).

c. $\displaystyle\int_C (xy+1)\,ds$

Here C is the lower semicircle $y = -\sqrt{9-x^2}$, traversed in the positive direction.

This integral is like (3). Notice that C be parametrized by the vector equation

$$\mathbf{r}(t) = t\,\mathbf{i} - \sqrt{9-t^2}\,\mathbf{j}, \quad -3 \le t \le 3$$

Then

$$\mathbf{r}'(t) = \mathbf{i} - \frac{t}{\sqrt{9-t^2}}\,\mathbf{k}$$

and

$$|\mathbf{r}'(t)| = \frac{3}{\sqrt{9-t^2}}$$

So by (3)

$$\begin{aligned}\int_C (xy+1)\,ds &= \int_{-3}^{3} (1 - t\sqrt{9-t^2})\frac{3}{\sqrt{9-t^2}}\,dt \\ &= -3\int_{-3}^{3} t\,dt + \int_{-3}^{3} \frac{3}{\sqrt{9-t^2}}\,dt \\ &= 0 + 3\sin^{-1}\frac{t}{3}\Big|_{-3}^{3} \\ &= 3\pi\end{aligned}$$

d. Let $\mathbf{F} = (x - z)\,\mathbf{i} + x\,\mathbf{k}$ be a velocity field flowing through a region in space and let C be the smooth curve defined by the vector equation $\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{k},\ 0 \le t \le 2\pi/3$. Find the flow along C in the direction of increasing t.

This is like (4). Now

$$\mathbf{F}(\mathbf{r}(t)) = (\cos t - \sin t)\,\mathbf{i} + \cos t\ \mathbf{k}$$

$$\frac{d\mathbf{r}}{dt} = -\sin t\ \mathbf{i} + \cos t\ \mathbf{k}$$

Hence

$$\mathbf{F} \cdot d\mathbf{r} = (1 - \sin t\ \cos t)\,dt$$

So that

$$\begin{aligned} \text{Flow} &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^{2\pi/3} (1 - \sin t \cos t)\,dt \\ &= \left(t - \frac{\sin^2 t}{2}\right) \Bigg|_0^{2\pi/3} \\ &= \frac{2\pi}{3} - \frac{3}{8} \end{aligned}$$