15.2 Iterated Integrals and Fubini's Theorem

Theorem 1. Fubini's Theorem

If $f(x, y)$ is continuous over the rectangular region $R:\ a \le x \le b,\ c \le y \le d$, then

$$\iint_R f(x,y)\,dA = \int_a^b \underbrace{\left(\int_c^d f(x,y)\,dy\right)}_{\text{a function of } x} dx \tag{1}$$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx \tag{2}$$

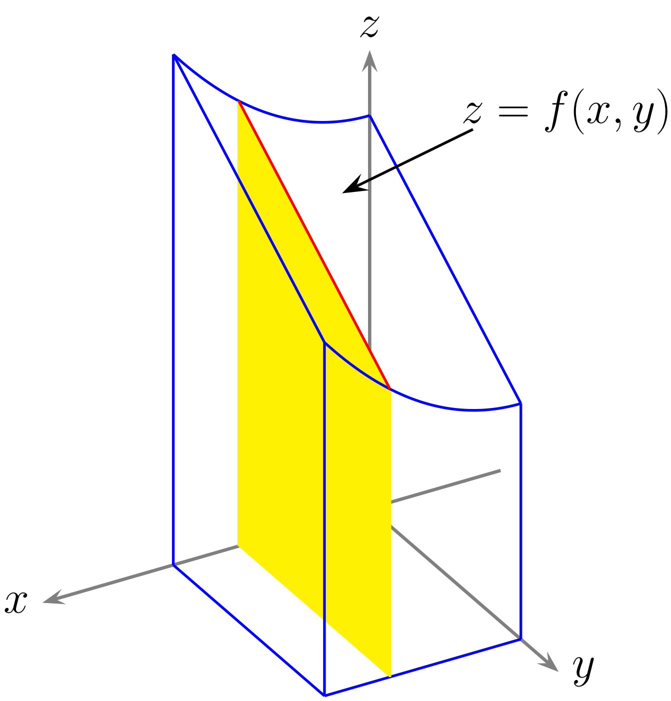
or

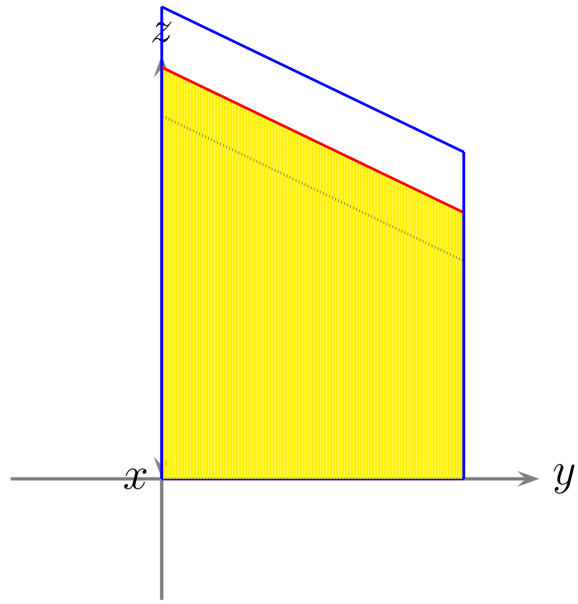
$$\iint_R f(x,y)\,dA = \int_c^d \underbrace{\left(\int_a^b f(x,y)\,dx\right)}_{\text{a function of } y} dy \tag{3}$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy \tag{4}$$

Remark. The expressions on the right are called **iterated** integrals. So Fubini's Theorem allows us to rewrite a double integral into a form that is more manageable as we see in the examples below. Notice that parenthesis surrounding the "inner integral" are normally not needed.

Motivation for Fubini's Theorem. Let's revisit a topic from last time. Find the volume below the surface $z = f(x, y)$ over the region R: $0 \leq x \leq b,\ 0 \leq y \leq d$.





For a fixed x_0, find the cross-sectional area shown.

So must consider the function $g(y) = f(x_0, y)$. It follows that the area of the cross section is

$$A(x_0) = \int_{y=0}^{y=d} g(y)\, dy$$

$$= \int_{y=0}^{y=d} f(x_0, y)\ dy$$

It follows from calculus II that the volume of the region below the surface is

$$\text{Volume} = \int_{x=0}^{x=b} A(x)\,dx$$
$$= \int_{x=0}^{x=b} \int_{y=0}^{y=d} f\left(x, y\right)\,dy\,dx$$

A similar argument would show that the volume is also given by

$$\text{Volume} = \int_{y=0}^{y=d} A(y)\,dy$$
$$= \int_{y=0}^{y=d} \int_{x=o}^{x=b} f\left(x, y\right)\,dx\,dy$$

Example 1. Evaluating Double Integrals

a. Let $R\colon 0 \leq x \leq 3,\ -2 \leq y \leq 0$. Sketch the region R and evaluate the double integral below.

$$\iint_R (5x^2y - 6xy)\, dA$$

b. Evaluate the double integral.

$$\int_0^{\pi/6} \int_0^1 3x \sin 2xy \, dx dy$$

Theorem 2. Fubini's Theorem (Stronger Form)

If $f(x, y)$ is continuous over a region R.

1. If R is defined by $a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)$, then

$$\iint_R f(x, y)\, dA \ = \int_a^b \int_{y=g_1(x)}^{y=g_2(x)} f(x, y)\, dy\, dx$$

2. If R is defined by $c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)$, then

$$\iint_R f(x, y)\, dA \ = \int_c^d \int_{x=h_1(y)}^{x=h_2(y)} f(x, y)\, dx\, dy$$

Example 2. Evaluating Double Integrals

Evaluate the following integrals.

a. $\displaystyle\int_0^1 \int_0^{\sqrt{1-s^2}} 8t\,dt\,ds$

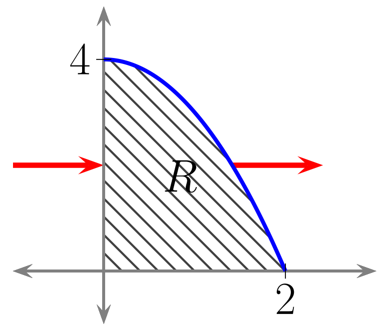
b. $\displaystyle\int_0^1 \int_{1-x}^{1-x^2} dy\,dx$

c. $\displaystyle\int_0^{\ln 2} \int_{e^y}^{2} dx\,dy$

d. $\displaystyle\int_0^3 \int_{\sqrt{x/3}}^{1} e^{y^3}\,dy\,dx$

Example 3. Sketch the region of integration and evaluate the integral below.

(5) $$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dy\, dx = I$$



As written, the integral in (5) seems impossible to evaluate. So we try to change the order of integration. Notice that the region R can also be described by

$$0 \le x \le \sqrt{4-y}, \quad 0 \le y \le 4$$

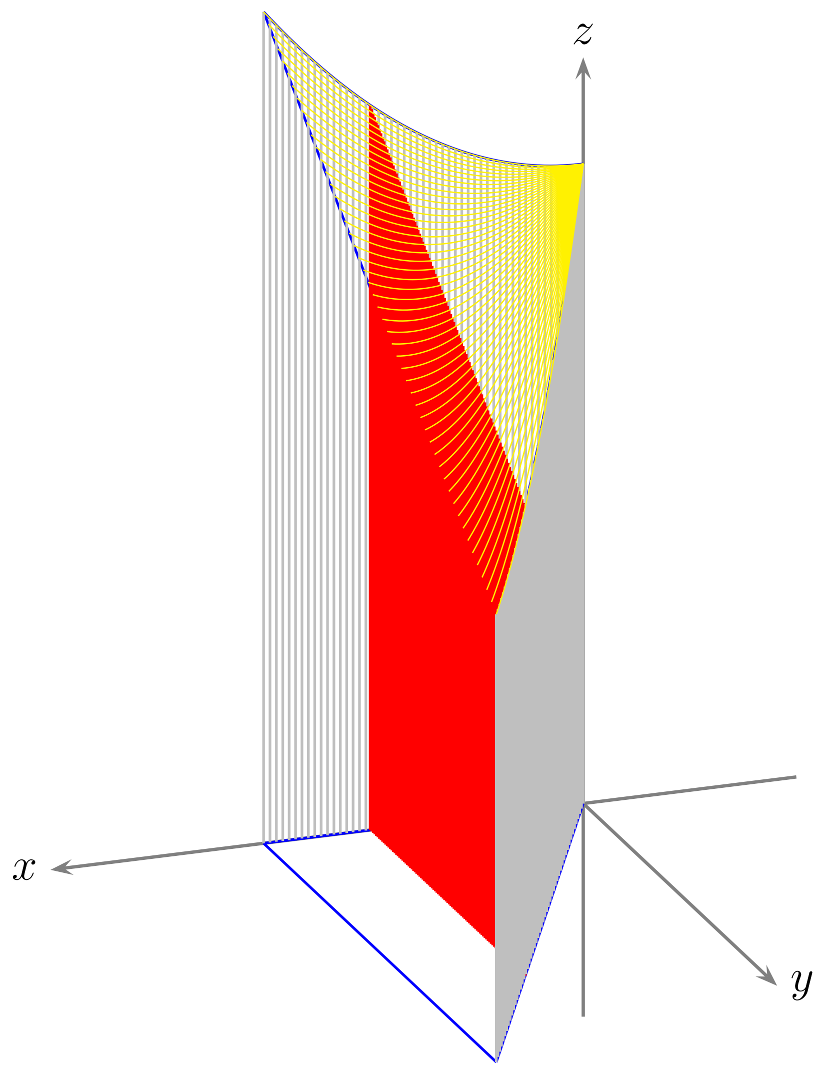
It follows that the iterated integral in (5) can be rewritten as

(6) $$\begin{aligned} I &= \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\, dx\, dy \\ &= \int_0^4 \frac{e^{2y}}{4-y} \int_0^{\sqrt{4-y}} x\, dx\, dy \\ &= \frac{1}{2} \int_0^4 \frac{e^{2y}}{4-y} \left(x^2 \Big|_0^{\sqrt{4-y}} \right) dy \\ &= \frac{1}{2} \int_0^4 \frac{e^{2y}}{4-y} \frac{4-y}{1}\, dy = \frac{1}{2} \int_0^4 e^{2y}\, dy \\ &= \frac{e^8 - 1}{4} \end{aligned}$$

Notice that Wolfram Alpha can correctly evaluate the integral in (5) and the equivalent integral in (6).

Example 4. Finding the volume below the surface $z = f(x, y)$.

Let $f(x, y) = x^2 + 3y$ be defined over the region $R\colon 0 \le x \le 3$ and $0 \le y \le 2x$. Find the volume beneath the surface $z = f(x, y)$.



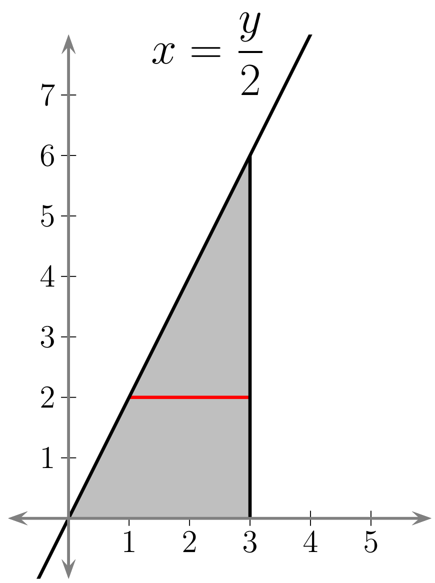
We try two methods.

Method 1: Integrate first with respect to y.

So the volume is given by the double integral:

$$\begin{aligned}
\text{Volume} &= \iint_R f(x,y)\,dA \\
&= \int_0^3 \int_0^{2x} (x^2 + 3y)\,dy\,dx \\
&= \int_0^3 \left(x^2 y + \frac{3y^2}{2} \right) \Bigg|_{y=0}^{y=2x} dx \\
&= \int_0^3 (2x^3 + 6x^2)\,dx \\
&= \cdots \\
&= \frac{189}{2}
\end{aligned}$$

Method 2: Integrate with respect to x.



$$\text{Volume} = \iint_R f(x,y)\,dA$$

$$= \int_0^6 \int_{y/2}^3 (x^2 + 3y)\,dx\,dy$$

$$= \int_0^6 \left(\frac{x^3}{3} + 3xy\right) \Bigg|_{x=y/2}^{x=3} dy$$

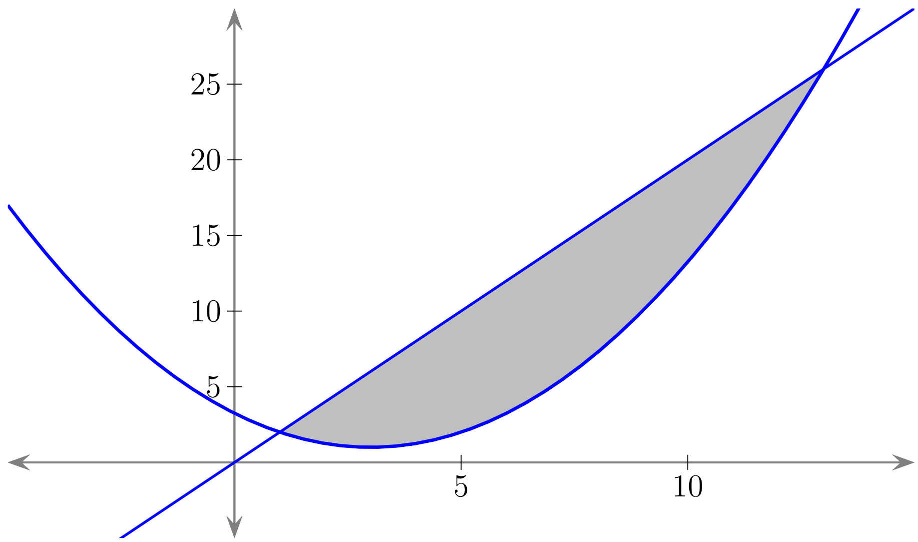
$$= \int_0^6 \left(-\frac{y^3}{24} - \frac{3y^2}{2} + 9y + 9\right) dy$$

$$= \cdots$$

$$= \frac{189}{2}$$

Example 5. Finding the area of a planar region.

Find the area of the region R bounded by the curves $y = (x-3)^2/4 + 1$ and $y = 2x$.



$$\text{Area} = \iint_R dA$$

$$= \int_1^{13} \int_{(x-3)^2/4+1}^{2x} dy\, dx$$

$$= \vdots$$

$$= 72$$

Average Value of a Function

As we did in Calculus I, we have the following definition.

Definition. Let f be an integrable function defined over a region R in the plane. Then the **average value of f over R** is given by

$$f_{\text{avg}} = \frac{1}{\text{area of } R} \iint_R f\, dA$$

Example 6. Finding the average value of a function.

Let $f(x, y) = 2x$. Find the average value of f over the region R from the previous example. We have

$$\begin{aligned} f_{\text{avg}} &= \frac{1}{\text{area of } R} \iint_R f\, dA \\ &= \frac{1}{72} \int_1^{13} \int_{(x-3)^2/4+1}^{2x} 2x\, dy\, dx \end{aligned}$$