16.6 Introduction to Parametric Surfaces

Parametric Surfaces

In this section we study the vector valued function $\mathbf{r}(u, v)$ of two parameters u and v. So let

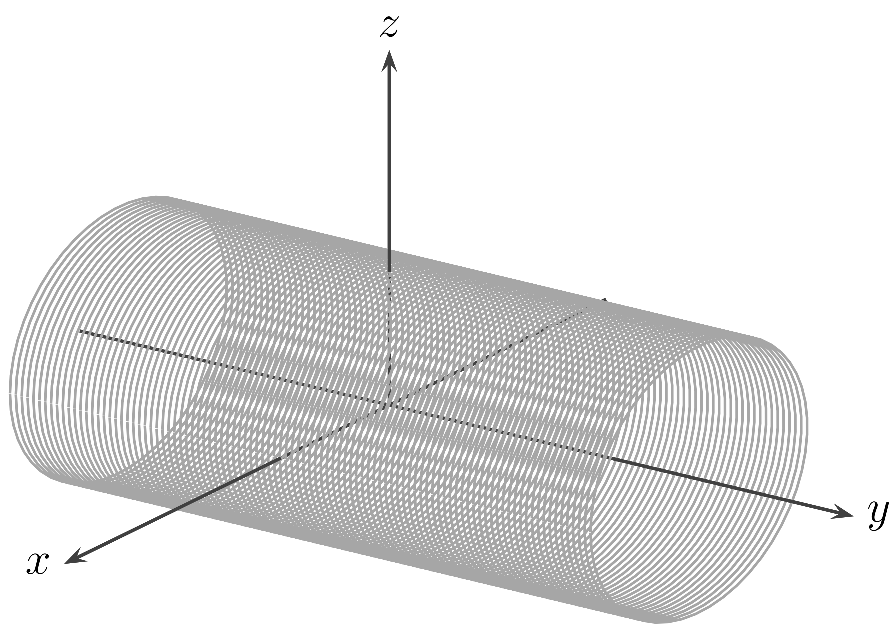
(1) $$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

defined on a region D of the so-called uv-plane.

The set of points $(x, y, z) \in \mathbb{R}^3$ with

(2) $$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad (u, v) \in D$$

is called a **parametric surface** S and the equations (2) are called the parametric equations of S.



Example 1. Identify and sketch the surface whose vector equation is

$$\mathbf{r}(u,v) = \cos u\,\mathbf{i} + v\,\mathbf{j} + \frac{3\sin u}{4}\,\mathbf{k}$$

The corresponding parametric equations are

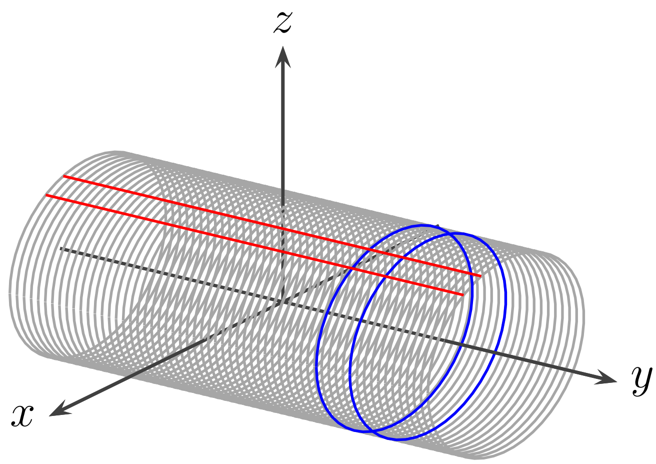
$$x = \cos u, \quad y = v, \quad z = \frac{3\sin u}{4}$$

Notice that

$$9x^2 + 16z^2 = 9\cos^2 u + 9\sin^2 u = 9$$

So that cross-sections parallel to the xz-plane are ellipses. Since $y = v$ without restriction, we obtain an elliptical cylinder parallel to the y-axis.

Suppose now that we fix $u = u_0$. Then $\mathbf{r}_1(v) = \mathbf{r}(u_0, v)$ is a vector-valued function of a single parameter v. Similarly, $\mathbf{r}_2(u) = \mathbf{r}(u, v_0)$ is a vector-valued function of the single parameter u. In each case, we generate families of *space curves* that lie on the surface S. A few of these surface curves are shown on the surface below (from the previous example).



It turns out to be very straightforward to find the parametric representation for a given surface of the form $z = f(x, y)$.

Example 2. Find the parametric representation of the paraboloid $z = x^2 + y^2 + 1$.

We give two representations.

The Easy One: Here we let $x = x$ and $y = y$. Then $z = x^2 + y^2 + 1$ so that

$$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + (x^2 + y^2 + 1)\,\mathbf{k}$$

The More Useful Representation (perhaps): For this one we work with the polar parameters r and θ. So let $x = r\cos\theta$ and $y = r\sin\theta$. It follows that $z = r^2 + 1$ so that

$$\mathbf{r}(r, \theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + (r^2 + 1)\,\mathbf{k}$$

Example 3. Can you describe the surface defined by the vector equation

$$\mathbf{r}(\phi, \theta) = a\sin\phi\cos\theta\,\mathbf{i} + a\sin\phi\sin\theta\,\mathbf{j} + a\cos\phi\,\mathbf{k}$$

for some $a > 0$? Here $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$.

Parametric Surfaces and Tangent Planes

Example 4. Find the equation of the tangent plane for the surface defined by the vector equation at $P_0 = P_0(-6, 1, 8)$.

$$S : \ \mathbf{r}(s,t) = \langle 2s^2 - t^3, s, 4t \rangle$$

Notice that the $\mathbf{r}(1,2) = P_0$. Now what can we say about the parametric curves $\mathbf{r}_a(s) = \mathbf{r}(s,2)$ and $\mathbf{r}_b(t) = \mathbf{r}(1,t)$?

Clearly, both curves lie on S and they intersect at P_0. Also, $\mathbf{r}_a'(1)$ is tangent to S at P_0 and $\mathbf{r}_b'(2)$ is tangent to S at P_0. It follows that $\mathbf{r}_a'(1) \times \mathbf{r}_b'(2)$ is orthogonal to the surface S at P_0. But

$$\mathbf{r}_a'(1) = \mathbf{r}_x(1,2) = \langle 4, 1, 0 \rangle$$
$$\mathbf{r}_b'(2) = \mathbf{r}_y(1,2) = \langle -12, 0, 4 \rangle$$

In particular,

$$\mathbf{r}_x(1,2) \times \mathbf{r}_y(1,2) = \langle 4, -16, 12 \rangle$$

It follows that the equation of the tangent plane at P_0 is given by

$$4(x+6) - 16(y-1) + 12(z-8) = 0$$

Or, after dividing through by 4 and rearranging, we obtain

$$x - 4y + 3z = 14$$

Is there any way that we can confirm this result independently?

Example 5. Redo the previous example by recognizing S as the level surface of some function (of three variables).

Notice that

$$x = 2s^2 - \left(\frac{4t}{4}\right)^3 = 2y^2 - \frac{z^3}{64}$$

It follows that S is the level surface $f(x, y, z) = 0$ of the function $f(x, y, z) = 2y^2 - z^3/64 - x$. Following the recipe from section 14.4 we have

$$\begin{aligned} f_x &= -1 \\ f_y &= 4y \quad \Longrightarrow \quad f_y(P_0) = 4 \\ f_z &= -3z^2/64 \quad \Longrightarrow \quad f_z(P_0) = -3 \end{aligned}$$

and $\langle -1, 4, -3 \rangle$ is normal to the tangent plane $x - 4y + 3z = 14$, as expected.

Example 6. Let S be a sphere of radius 4 centered at the origin. Find the equation of the plane tangent S to a at $Q_0(3, \sqrt{3}, 2)$. In Example 3 we saw that S can be defined by the vector equation

$$S:\ \mathbf{r}(\phi, \theta) = 4\sin\phi\cos\theta\,\mathbf{i} + 4\sin\phi\sin\theta\,\mathbf{j} + 4\cos\phi\,\mathbf{k}$$

It is routine to show that $\mathbf{r}(\pi/3, \pi/6) = Q_0$. Now

$$\begin{aligned}\mathbf{r}_\phi &= \langle 4\cos\phi\cos\theta, 4\cos\phi\sin\theta, -4\sin\phi\rangle \\ \Longrightarrow \mathbf{r}_\phi(\pi/3, \pi/6) &= \left\langle \sqrt{3}, 1, -2\sqrt{3} \right\rangle \\ \mathbf{r}_\theta &= \langle -4\sin\phi\sin\theta, 4\sin\phi\cos\theta, 0\rangle \\ \Longrightarrow \mathbf{r}_\theta(\pi/3, \pi/6) &= \left\langle -\sqrt{3}, 3, 0 \right\rangle\end{aligned}$$

An easy calculation shows that

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \left\langle 6\sqrt{3}, 6, 4\sqrt{3} \right\rangle$$

It follows that the equation of the tangent plane at Q_0 is given by

$$6\sqrt{3}(x-3) + 6(y-\sqrt{3}) + 4\sqrt{3}(z-2) = 0$$

Why is the last example easy to check independently?