14.6 Directional Derivatives and the Gradient

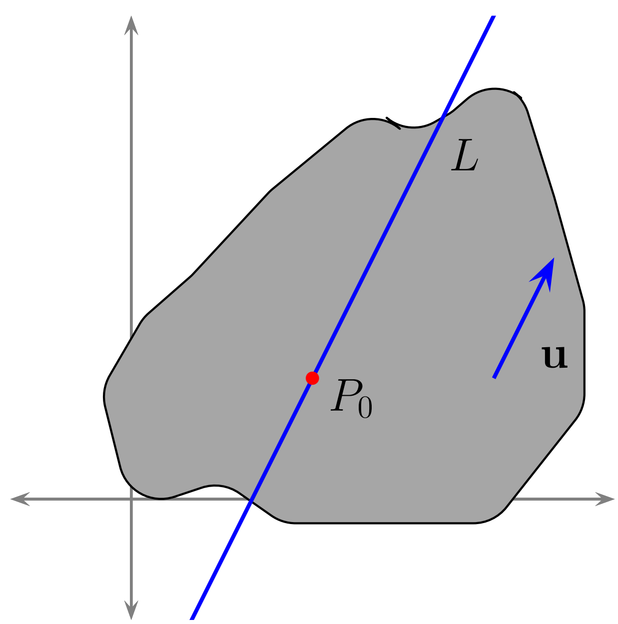
Directional Derivatives in the Plane

Suppose that $f(x, y)$ is a differentiable function. Then the rate at which f changes with respect to t along a differentiable curve $x = g(t),\ y = h(t)$ is given by the chain rule as

(1)
$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Now suppose that $P_0\,(x_0, y_0)$ lies in an open region R of the plane (as shown) and that $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$ is a **unit** vector. Now let L be the line passing through P_0 in the direction $\mathbf{u}$. Then a parameterization of L is given by

$$L:\ x = x_0 + s\,u_1, \qquad y = y_0 + s\,u_2$$



Remark. Notice that s measures arc length from P_0 in the direction $\mathbf{u}$, i.e., s is the arc-length parameter. To see this note that

$$\sqrt{(s\,u_1)^2 + (s\,u_2)^2} = s\sqrt{(u_1)^2 + (u_2)^2} = s$$

since $\mathbf{u}$ is a unit vector.

Now we wish to find the rate of change in the direction $\mathbf{u}$, we need to calculate df/ds at P_0. We have

Definition. Directional Derivative

The derivative of f at P_0 in the direction of the unit vector $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$ is

(2) $$\left(\frac{df}{ds}\right)_{\mathbf{u},\,P_0} = \lim_{s \to 0^+} \frac{f\,(x_0 + s\,u_1, y_0 + s\,u_2) - f\,(x_0, y_0)}{s}$$

provided that the limit exists.

Remark. The directional derivative is also denoted by $(D_{\mathbf{u}} f)_{P_0}$.

Example 1. Directional Derivatives

Let $f(x, y)$ be a function. Use equation (2) to find $(D_{\mathbf{i}}f)_{(x_0,y_0)}$.

$$\begin{aligned}(D_{\mathbf{i}}f)_{(x_0,y_0)} &= \lim_{s\to 0^+} \frac{f\left(x_0+s, y_0\right)-f\left(x_0, y_0\right)}{s} \\ &= \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}\end{aligned}$$

In other words, the *directional derivative* in the $\mathbf{i}$ direction is the partial derivative with respect to x, as we mentioned last time. In symbols,

$$\frac{\partial f}{\partial x} = D_{\mathbf{i}}f$$

Similarly,

$$\frac{\partial f}{\partial y} = D_{\mathbf{j}}f$$

Example 2. Finding a Directional Derivative

Let $f(x, y) = 2xy + y$ and let $\mathbf{u} = \frac{1}{2}\,\mathbf{i} + \frac{\sqrt{3}}{2}\,\mathbf{j}$. Use equation (2) to find $(D_{\mathbf{u}}f)_{(1,2)}$.

$$\begin{aligned}
(D_{\mathbf{u}}f)_{(1,2)} &= \lim_{s\to 0^+} \frac{f\left(1 + s/2, 2 + s\sqrt{3}/2\right) - f(1,2)}{s} \\
&= \lim_{s\to 0^+} \frac{2\left(1 + s/2\right)\left(2 + s\sqrt{3}/2\right) + \left(2 + s\sqrt{3}/2\right) - (2(1)(3) + (2))}{s} \\
&= \lim_{s\to 0^+} \frac{4 + s\sqrt{3} + 2s + s^2\left(\sqrt{3}/2\right) + 2 + s\sqrt{3}/2 - 6}{s} \\
&= \lim_{s\to 0^+} \frac{s\left(\sqrt{3} + 2 + s\sqrt{3}/2 + \sqrt{3}/2\right)}{s} \\
&= \lim_{s\to 0^+} \left(\sqrt{3} + 2 + s\sqrt{3}/2 + \sqrt{3}/2\right) \\
&= \frac{3\sqrt{3} + 4}{2}
\end{aligned}$$

Remark. Observe that the directional derivative is a **scalar**. In fact, the directional derivative gives the *instantaneous rate of change* in the $\mathbf{u}$ direction.

The Directional Derivative and the Gradient

Let f be a differentiable function, $P = (x, y)$, and $P_0 = P_0\,(x_0, y_0)$ and let $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$ be a unit vector.

Now since f is differentiable, the limit below exists along any path to (x_0, y_0).

(3) $$0 = \lim_{P \to P_0} \frac{f\,(x, y) - f\,(x_0, y_0) - f_x\,(x_0, y_0)\,(x - x_0) - f_y\,(x_0, y_0)\,(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}$$

In particular, the limit exists along the path

(4) $$x = x_0 + s\,u_1,\ y = y_0 + s\,u_2 \text{ as } s \to 0^+$$

By combining (4) into (3), we obtain

$$0 = \lim_{s\to 0^+} \left\{ \frac{f(x_0 + s\,u_1, y_0 + s\,u_2) - f(x_0, y_0)}{s} - \frac{f_x(x_0, y_0)(su_1) - f_y(x_0, y_0)(su_2)}{s} \right\}$$

$$= \lim_{s\to 0^+} \frac{f(x_0 + s\,u_1, y_0 + s\,u_2) - f(x_0, y_0)}{s} - \lim_{s\to 0^+} \frac{f_x(x_0, y_0)(su_1) - f_y(x_0, y_0)(su_2)}{s}$$

$$= (D_{\mathbf{u}}f)_{P_0} - \lim_{s\to 0^+} \frac{f_x(x_0, y_0)(u_1) - f_y(x_0, y_0)(u_2)}{1}\,\frac{s}{s}$$

$$= (D_{\mathbf{u}}f)_{P_0} - f_x(x_0, y_0)\,u_1 - f_y(x_0, y_0)\,u_2$$

In other words,

$$(D_{\mathbf{u}}f)_{P_0} = f_x(x_0, y_0)\,u_1 + f_y(x_0, y_0)\,u_2$$

$$= \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2$$

$$= \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)_{P_0} \mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \mathbf{j}\right]}_{\nabla f} \cdot \underbrace{(u_1\,\mathbf{i} + u_2\,\mathbf{j})}_{\mathbf{u}}$$

Definition. The Gradient Vector

The **gradient vector** of $f(x, y)$ at $P_0\,(x_0, y_0)$ is the vector

(5)
$$\nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}$$

In 3 dimensions we make the obvious adjustments. That is, the gradient vector of $f(x, y, z)$ at $P_0(x_0, y_0, z_0)$ is the vector

(6)
$$\nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}$$

The gradient vector is usually just called the **gradient** and is denoted by

∇f or **grad** f or **del** f

Remark. Please review the significance of the gradient in the text on page 966.

Theorem 1. The Directional Derivative is a Dot Product

A careful inspection of the calculations above yields the following nifty formula.

(7)
$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}$$

Example 3. Directional Derivatives using (7).

a. Verify the result from the previous example.
For $f(x,y) = 2xy + y$ with $\mathbf{u} = \frac{1}{2}\,\mathbf{i} + \frac{\sqrt{3}}{2}\,\mathbf{j}$ find $(D_{\mathbf{u}}f)_{(1,2)}$.
Now

$$\nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}$$

$$\implies \nabla f = 2y\,\mathbf{i} + (2x+1)\,\mathbf{j}$$

hence

$$(\nabla f)_{(1,2)} = 4\,\mathbf{i} + 3\,\mathbf{j}$$

Thus

$$(D_{\mathbf{u}}f)_{(1,2)} = (4\,\mathbf{i} + 3\,\mathbf{j}) \cdot \left(\frac{1}{2}\,\mathbf{i} + \frac{\sqrt{3}}{2}\,\mathbf{j}\right)$$

$$= \frac{4}{2} + \frac{3\sqrt{3}}{2}$$

as we saw before.

b. Let $g(x,y) = xy + 2\sin y$ and let $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$. Find each of the following.

i. $D_{\mathbf{u}}g$

$g_x = y$ and $g_y = x + 2\cos y$ so that

$$\nabla g = y\,\mathbf{i} + (x + 2\cos y)\,\mathbf{j}$$

Thus

$$\begin{aligned} D_{\mathbf{u}}g &= \nabla g \cdot \mathbf{u} \\ &= y\cos\theta + (x + 2\cos y)\sin\theta \end{aligned}$$

ii. Find $D_{\mathbf{u}}g$ for $\theta = \pi/3$.

From part (i) we have

$$\begin{aligned} D_{\mathbf{u}}g &= y\cos\theta + (x + 2\cos y)\sin\theta \\ &= y\cos(\pi/3) + (x + 2\cos y)\sin(\pi/3) \\ &= y/2 + \sqrt{2}\,(x + 2\cos y)\,/2 \end{aligned}$$

The Direction of Most Rapid Change

Notice that by Theorem 1 and definitions from chapter 12, we have

$$\begin{aligned} D_{\mathbf{u}}f &= (\nabla f) \cdot \mathbf{u} \\ &= |\nabla f| \; |\mathbf{u}| \; \cos\theta \\ &= |\nabla f| \; \cos\theta \end{aligned}$$

where θ is the angle between ∇f and $\mathbf{u}$. It follows that if $\nabla f \neq \mathbf{0}$ then

1. The function increases most rapidly when $\cos\theta = 1$, i.e., when $\mathbf{u}$ is in the direction of the gradient ∇f. In this case, the directional derivative is equal to $|\nabla f|$.

2. The function decreases the most rapidly in the direction of $-\nabla f$. In this case the directional derivative is $-|\nabla f|$.

3. Any direction $\mathbf{u}$ orthogonal to a nonzero gradient is a direction of zero change for if $\theta = \pi/2$, then

$$D_{\mathbf{u}}f = |\nabla f| \cos(\pi/2) = 0$$

Example 4. Using the Gradient

Let $f(x,y) = \dfrac{x^2}{y-x}$.

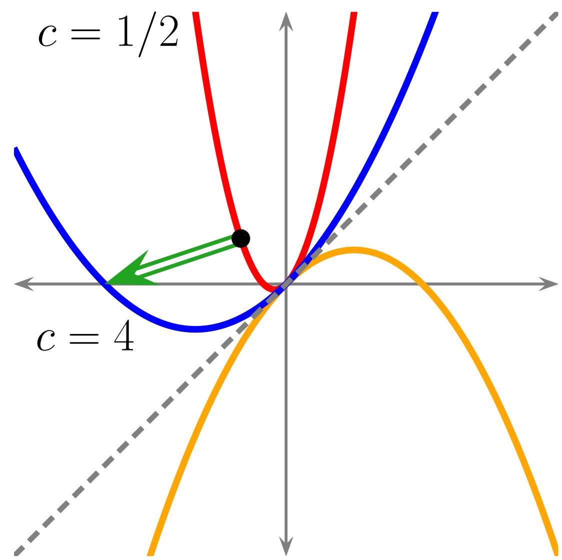
a. Sketch the level curves ($f(x,y) = c$) of f. (Use $c = -3,\ 1/2,$ and 4.)

For example, if $c = \dfrac{1}{2}$

$$\implies \frac{x^2}{y-x} = \frac{1}{2}$$

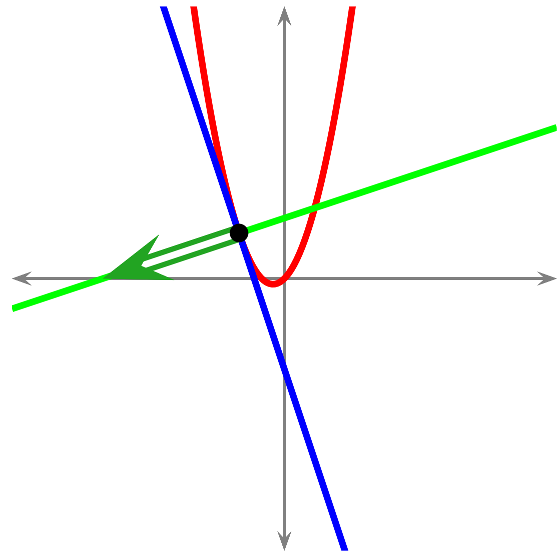
$$\implies y = 2x^2 + x$$

which is shown in red.



By examining the level curves, our intuition suggests that the direction of maximal change at $(-1, 1)$ should be roughly in the direction shown. Why?

b. Find the direction of maximal change at the point $(-1, 1)$.



From calculus, we know that the slope of the tangent line at $(-1, 1)$ is

$$\left.\frac{dy}{dx}\right|_{x=-1} = (4x+1)\Big|_{x=-1}$$
$$= -3$$

It follows that the slope of the normal is

$$m^{\perp} = \frac{-1}{3}$$

So we expect the gradient through $(-1, 1)$ to be the same as the **normal** (shown above in green). Now

$$f_x = \frac{2x(y-x) - x^2(-1)}{(y-x)^2}$$
$$= \frac{2xy - x^2}{(y-x)^2} \qquad \Longrightarrow f_x(-1,1) = \frac{-3}{4}$$

and

$$f_y = \frac{-x^2}{(y-x)^2} \qquad \Longrightarrow f_y(-1,1) = \frac{-1}{4}$$

Hence

$$(\nabla f)_{(-1,1)} = \frac{-3}{4}\,\mathbf{i} - \frac{1}{4}\,\mathbf{j}$$

as expected.

Remark. Technically, we should find a unit vector since the question did ask us for the *direction*. So the direction of maximal increase is the *unit vector*

$$\frac{1}{\sqrt{10}}\,(-3\,\mathbf{i} - \mathbf{j})$$

c. What is the direction of **zero** change at $(-1, 1)$?

Clearly, it is $\frac{1}{\sqrt{10}}\,(\mathbf{i} - 3\,\mathbf{j})$ or its negative.

More About Level Curves and the Gradient

Suppose that $f(x, y) = c$ along a **smooth curve** $\mathbf{r} = g(t)\,\mathbf{i} + h(t)\,\mathbf{j}$ (so $\mathbf{r}(t)$ is a level curve of f), then $f\left(g(t), h(t)\right) = c$. If f is differentiable we may differentiate both sides so that

$$\begin{aligned} \frac{d}{dt}\, f\left(g(t), h(t)\right) &= \frac{d}{dt}\, c \\ \Longrightarrow \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} &= 0 \end{aligned}$$

It follows that

$$\left(\frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}\right) \cdot \left(\frac{dg}{dt}\,\mathbf{i} + \frac{dh}{dt}\,\mathbf{j}\right) = 0$$

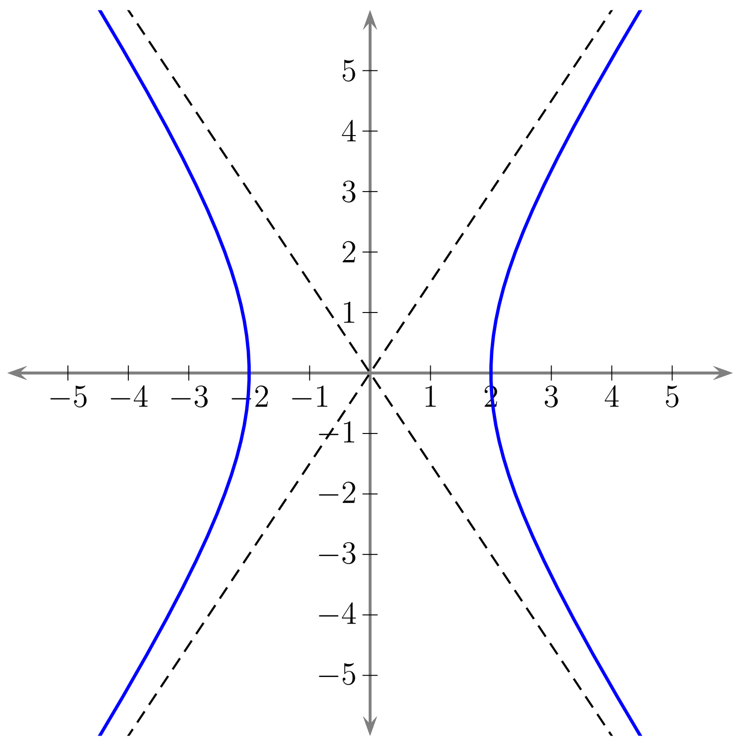
or

$$\nabla f \cdot \frac{d\mathbf{r}}{dt} = 0$$

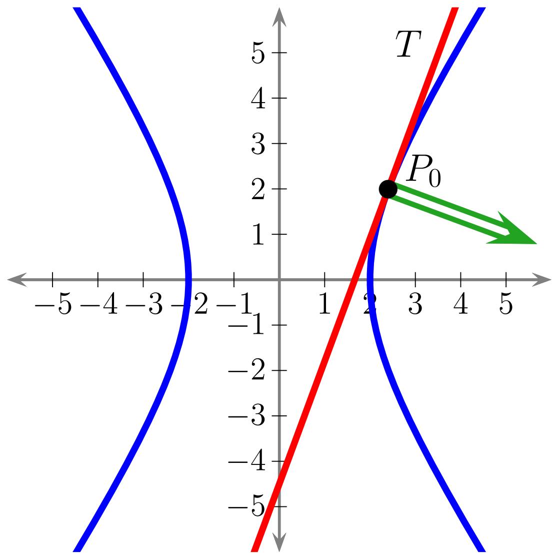
In other words, ∇f is orthogonal to the level curve $f\left(g(t), h(t)\right) = c$ at every point of the domain of the differentiable function f.

Example 5. Find the equation of the tangent line to the hyperbola

$$\frac{x^2}{2^2} - \frac{y^2}{3^2} = 1 \tag{8}$$



Now let $f(x,y) = 9x^2 - 4y^2$. Then the hyperbola (8) is the (smooth) level curve $f(x,y) = 36$. Find the equation of the tangent line at $P_0 = \left(\sqrt{52}/3, 2\right)$.



Notice that if $\mathbf{n} = A\,\mathbf{i} + B\,\mathbf{j}$ is normal to the tangent line, T, at P_0, then the equation for T is

(9) $$0 = A\left(x - \frac{\sqrt{52}}{3}\right) + B\,(y-2)$$

But

$$\nabla f = 18x\,\mathbf{i} - 8y\,\mathbf{j} \quad \text{and}$$

$$(\nabla f)_{P_0} = 6\sqrt{52}\,\mathbf{i} - 16\,\mathbf{j}$$

So by (9)

$$T: \qquad 0 = 6\sqrt{52}\left(x - \frac{\sqrt{52}}{3}\right) - 16\,(y-2)$$