14.3 Partial Derivatives

The Derivative of a function of a Single Variable

We motivate the definition of the derivative of a function of two or more variables as follows. Recall from calculus I,

Definition. Derivatives in One Dimension

Let $f : D \subset \mathbb{R} \to \mathbb{R}$ and let x_0 be an interior point of D. Then f is differentiable at x_0 if the limit below exists.

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \tag{1}$$

and the number $f'(x_0)$ is called the derivative of f at x_0.

We also saw that although continuity didn't guarantee differentiability, the converse was true. That is, differentiability implies continuity.

Theorem 1. Differentiability Implies Continuity

If f is differentiable at x_0, then f is continuous at x_0.

Remark. Can you prove this?

It turns out that equation (1) can be restated.

Definition. Derivatives in One Dimension - Alternate Definition

Let $f : D \subset \mathbb{R} \to \mathbb{R}$ and let x_0 be an interior point of D. Then f is differentiable at x_0 if there is a number $f'(x_0)$ such that

(2)
$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - (x - x_0)f'(x_0)}{|x - x_0|} = 0$$

and the number $f'(x_0)$ is called the derivative of f at x_0.

We use this definition to create a definition of the derivative for functions of several variables.

Definition. Derivatives in Higher Dimensions

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ and let P_0 be an interior point of D. Then f is **differentiable** at P_0 if there is a vector $f'\left(P_0\right)$ such that

(3)
$$\lim_{P \to P_0} \frac{f\left(P\right) - f\left(P_0\right) - \left(\overrightarrow{P_0P}\right) \cdot f'\left(P_0\right)}{|P - P_0|} = 0$$

For the two-dimensional case, this becomes

Definition. Derivatives in Two Dimensions

Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ and let (x_0, y_0) be an interior point of D. Then f is **differentiable** at (x_0, y_0) if there are two numbers $f_1(x_0, y_0)$ and $f_2(x_0, y_0)$ such that

(4) $$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - f_1(x_0,y_0)(x-x_0) - f_2(x_0,y_0)(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

The vector $f_1(x_0, y_0)\,\mathbf{i} + f_2(x_0, y_0)\,\mathbf{j}$ is called the derivative of f at (x_0, y_0).

Let $$\begin{aligned} z &= L(x,y) \\ &= f(x_0,y_0) + f_1(x_0,y_0)(x-x_0) + f_2(x_0,y_0)(y-y_0), \end{aligned}$$

then (4) says that f is differentiable at (x_0, y_0) if there is a linear function $L(x, y)$ such that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - L(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

The equation

$$z = f(x_0, y_0) + f_1(x_0, y_0)(x - x_0) + f_2(x_0, y_0)(y - y_0)$$

is called the **tangent plane** of f at (x_0, y_0).

Now to find the $f_1(x_0, y_0)$ let $y = y_0$ in (4).

$$\lim_{(x,y_0)\to(x_0,y_0)} \frac{f(x, y_0) - f(x_0, y_0) - f_1(x_0, y_0)(x - x_0)}{|x - x_0|} = 0$$

Now let $h(x) = f(x, y_0)$ above to obtain

(5) $$\lim_{(x,y_0)\to(x_0,y_0)} \frac{h(x) - h(x_0) - f_1(x_0, y_0)(x - x_0)}{|x - x_0|} = 0$$

Now comparing this with (2) we see that $f_1(x_0, y_0) = h'(x_0)$. In other words, $f_1(x_0, y_0)$ is the derivative of $h(x) = f(x, y_0)$ at x_0. It is called the **partial derivative** of f with respect to x. Similarly, $f_2(x_0, y_0)$ is the partial derivative of f with respect to y.

It may be more beneficial to rewrite (5) in the usual manner. We have

Definition. Partial Derivatives

The partial derivative of $f(x, y)$ with respect to x at (x_0, y_0) is

$$\begin{aligned} \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} &= \frac{d}{dx} f\left(x, y_0\right) \Big|_{x=x_0} \\ &= \lim_{h\to 0} \frac{f\left(x_0 + h, y_0\right) - f\left(x_0, y_0\right)}{h} \end{aligned} \tag{6}$$

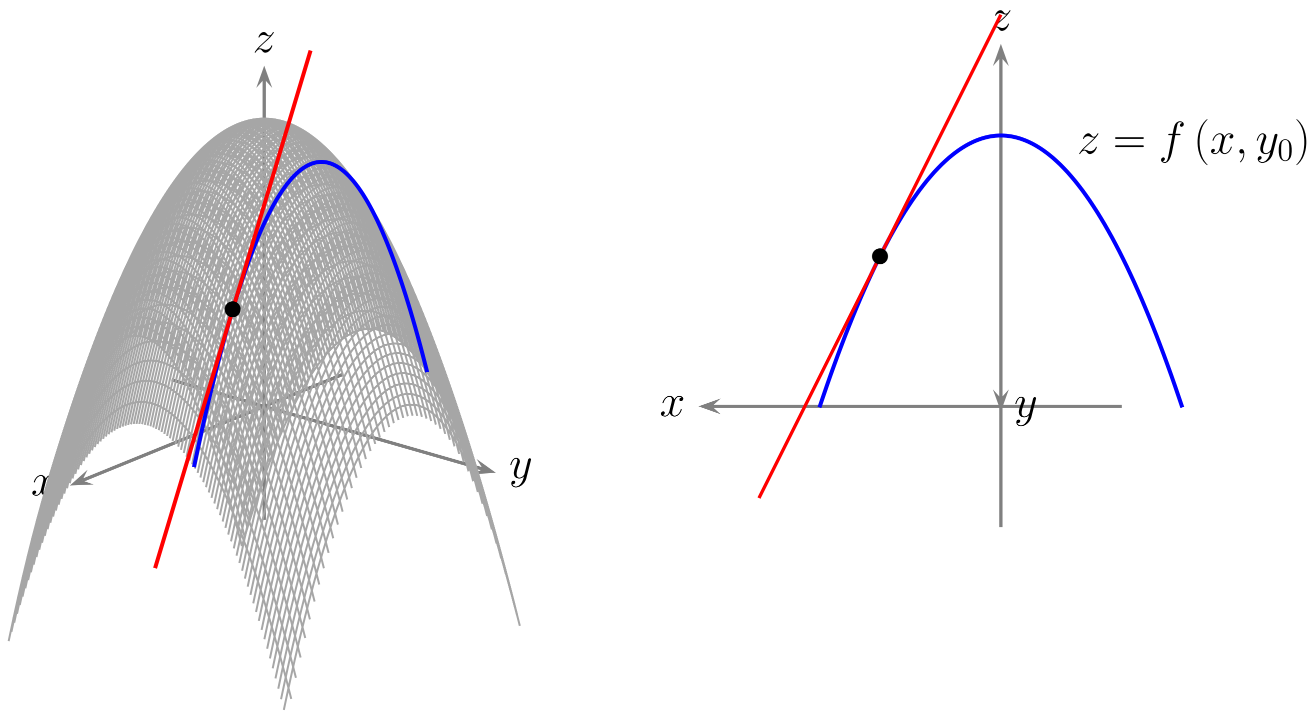
provided the limit exists.

Remark. Notice that $\dfrac{\partial f}{\partial x}$ gives the instantaneous rate of change in the $\mathbf{i}$ direction. The "partials" with respect to y and z are defined in a similar manner.

According to equation (6) we may compute the partial derivative of f with respect to x by taking the usual derivative of $f(x, y)$ (or $f(x, y, z)$) and treating the other variables as constants. We illustrate below.

More on Tangent Planes

Suppose that $z = f(x, y)$ is a surface and $(x_0, y_0) \in \text{Dom}(f)$. Then the vertical plane $y = y_0$ intersects the surface at curve $z = f\,(x, y_0)$.



Notice that the slope of the “tangent line” in the plane $y = y_0$ is given by

$$f_1\,(x_0, y_0) = \frac{\partial f}{\partial x}\bigg|_{(x_0, y_0)} = f_x\,(x_0, y_0)$$

Using the language of vectors, the tangent line is parallel to the vector $\mathbf{u} = \mathbf{i} + f_x\,(x_0, y_0)\,\mathbf{k}$. In a similar manner, the tangent line in the plane $x = x_0$ is parallel to the vector $\mathbf{v} = \mathbf{j} + f_y\,(x_0, y_0)\,\mathbf{k}$. Since both of these vectors are parallel to the tangent plane, we compute their cross-product to find the vector normal to the tangent plane. Thus, the vector normal to the tangent plane is

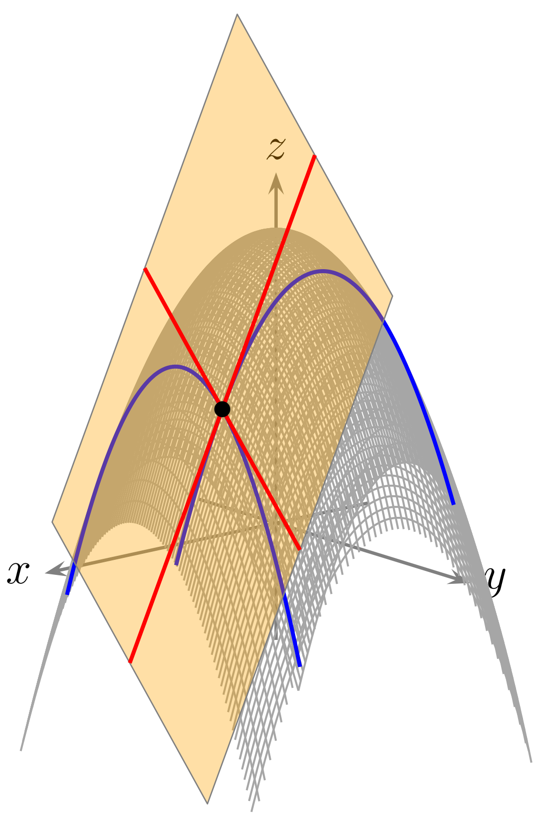
$$\mathbf{u} \times \mathbf{v} = -f_x(x_0, y_0)\,\mathbf{i} - f_y(x_0, y_0)\,\mathbf{j} + \mathbf{k}$$

It follows that the equation of the plane tangent to the surface $z = f(x, y)$ at (x_0, y_0) is given by

$$0 = -f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + (z - f(x_0, y_0))$$

or

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$



Example 1. Computing Partial Derivatives

Let $f(x,y) = \dfrac{x \sin y}{x + y^2}$. Compute the indicated partial derivatives.

a. Find $\left.\dfrac{\partial f}{\partial x}\right|_{(1,\pi/3)}$.

b. Find $\dfrac{\partial f}{\partial y} = \left.\dfrac{\partial f}{\partial y}\right|_{(x,y)}$.

Notice that the result of the second computation in the previous example was also a function of (x, y). This suggests we have an analogue to higher order derivatives.

Example 2. Higher-Order Partial Derivatives.

Let $g(x, y, z) = xz \cos y^2$. Compute the indicated partial derivatives.

a. Find $\dfrac{\partial g}{\partial x}$.

b. Find $\dfrac{\partial^2 g}{\partial y\,\partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial g}{\partial x}\right)$.

c. Find $\dfrac{\partial^2 g}{\partial x\,\partial y}$

The Mixed Derivative Theorem

You may have noticed that the mixed second-order partials in the last example (parts (b) and (c)) turned out to be equal. While this is not always the case, the following theorem tells us precisely when this is true.

Theorem 2. The Mixed Derivative Theorem

If $f(x, y)$ and its partial derivatives $f_x, f_y, f_{xy},$ and f_{yx} are defined throughout an open region containing (a, b) and are continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Theorem 3. Continuity of Partials Implies Differentiability

If the partial derivatives f_x and f_y of $f(x, y)$ are continuous throughout an open region R, then f is differentiable at every point in R.

Theorem 4. Differentiability Implies Continuity

If f is differentiable at P_0, f is contiuous at P_0.