14.1 Functions of Several Variables

For vector-valued functions of a **single** variable, it was often convenient to imagine that the functions described the movement of a particle along a curve in space and that the independent variable represented time.

In chapter this chapter we expand our domains to include functions of more than one variable. For example, the function $T = f(x, y, z)$ that yields the temperature of each point in this room. This a **real-valued** function that depends on the $x, y,$ and z coordinates of a point in space. More precisely we have

Definition. Functions of n Independent Variables

Suppose that D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns one and only one real number

$$w = f\left(x_1, x_2, \ldots, x_n\right)$$

to each element in D. The set D is called the **domain** of f and the set of all outputs is called the **range** of f.

Regions in the Plane and in Space

Although the domain of a function of a single (real) variable can be complicated, many of examples that we encountered in calculus are very straight-forward. For example, the domain of

$$f(x) = \frac{\sqrt{1-x}}{2x+1}$$

is

$$\left(-\infty, \frac{-1}{2}\right) \cup \left(\frac{-1}{2}, 1\right]$$

The situation can often be more complicated in higher dimensions.

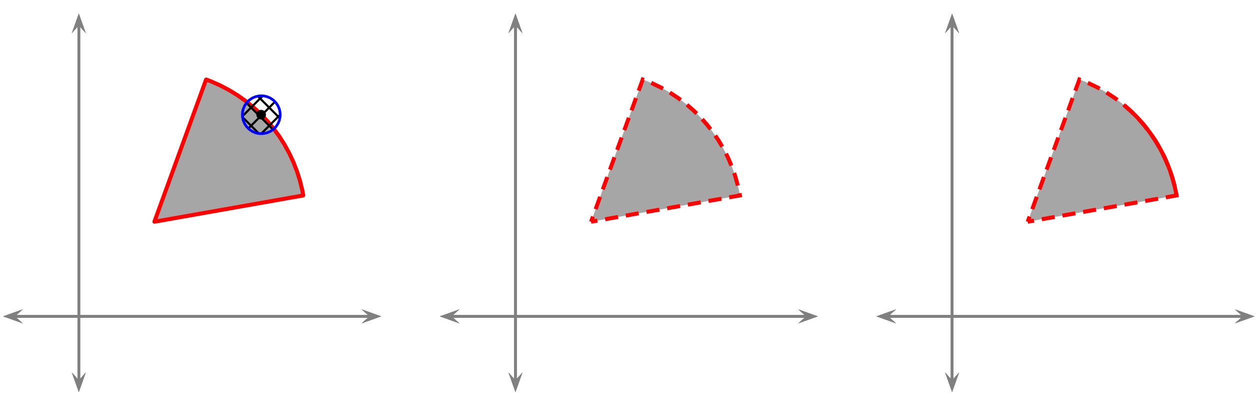
Definition. Let R be a region in the xy-plane. The point (x_0, y_0) is an **interior** point of R if there is a disk, D, centered at (x_0, y_0) such that $D \subset R$.

(x_0, y_0) is called a **boundary point** of R if *every* disk centered at (x_0, y_0) contains points that lie in R and outside of R.

The region R is said to be **closed** if R contains all of its boundary points. The region is called **open** if consists of its interior points only.

Remark. There are regions which are neither open nor closed.

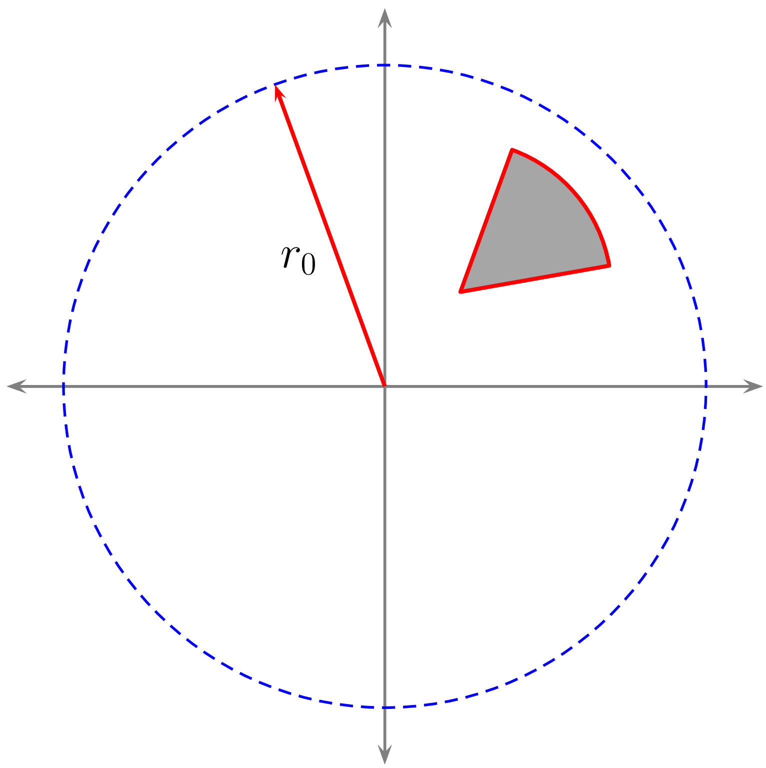
Example 1. **Regions in the Plane**



Definition. Bounded and Unbounded Regions in Space

A region in the plane is **bounded** if it can be placed inside a disk centered at the origin (of fixed radius). Otherwise, the region is called unbounded.

Each of the regions in the previous example is bounded. For example,



See the examples below for unbounded regions.

Level Curves

Definition. Graphs and Level Curves

The set of points in the plane where $f(x, y)$ is constant is called a **level curve** of f. The **graph** of f is the set of all ordered triples, $(x, y, f(x, y))$ such that (x, y) is in the domain of f. The graph of f is also called the **surface** $z = f(x, y)$.

Example 2. Sketching Level Curves

Let $f(x, y) = \dfrac{x^2}{y - x}$.

a. Find the domain of f.

$$\{(x,\, y) \mid x,\, y \in \mathbb{R},\ x \neq y\}$$

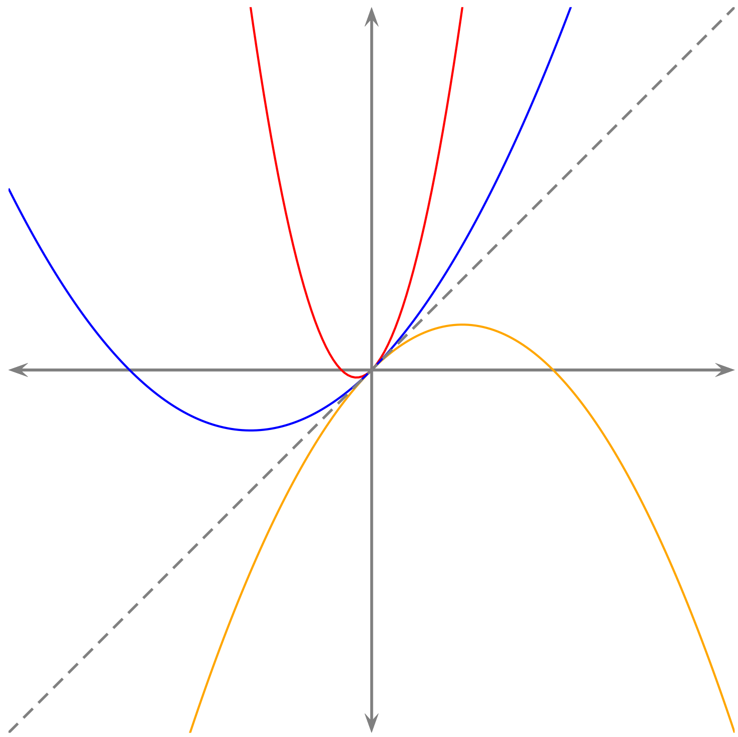
or just

$$x \neq y$$

b. Is the domain an open or closed region?

c. Is the domain bounded or unbounded?

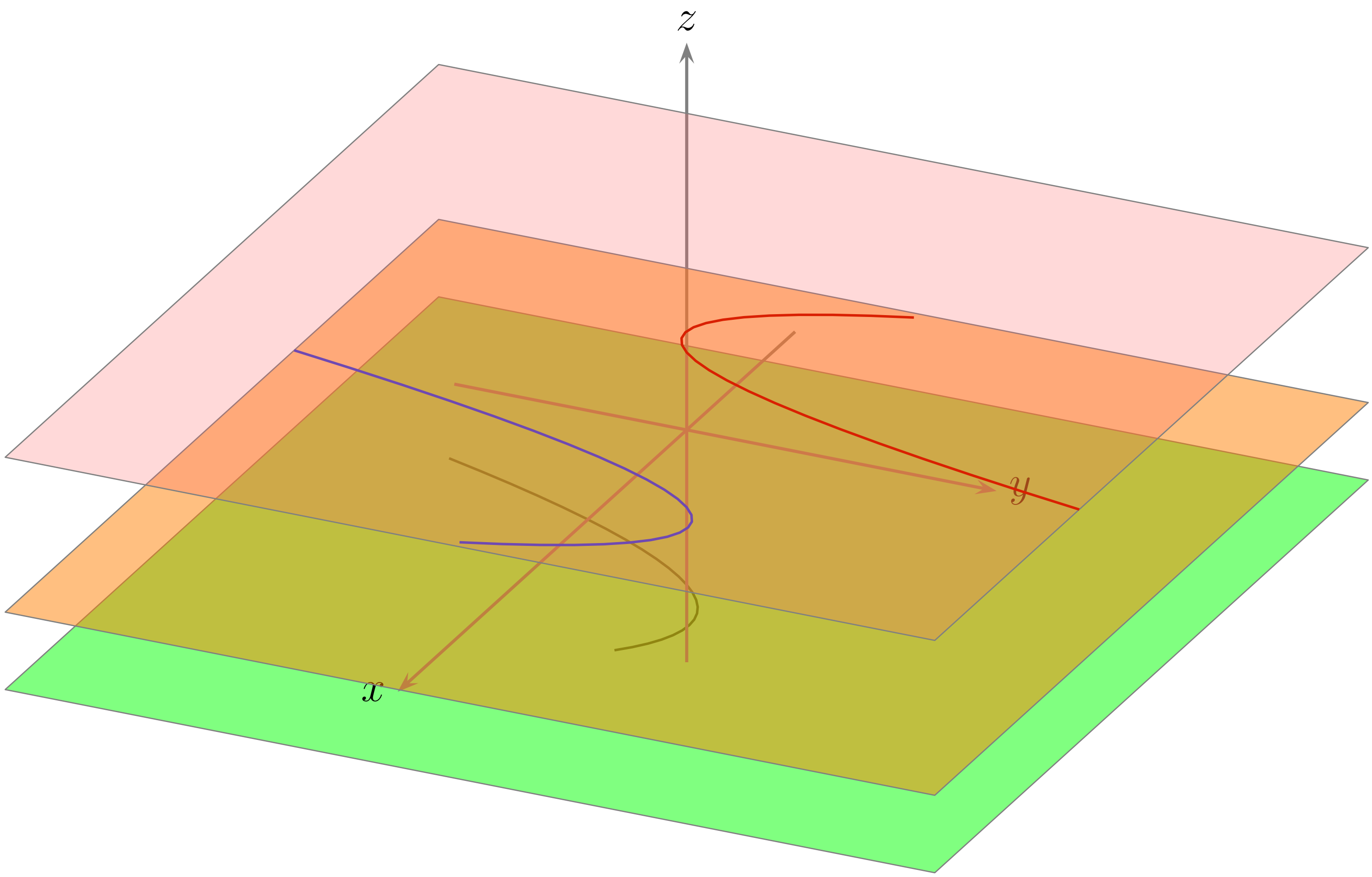
d. Sketch the level curves ($f(x, y) = c$) of f. (Use $c = -3,\ 1/2,$ and 4.)



For example, if $c = \dfrac{1}{2}$

$$\Longrightarrow \frac{x^2}{y-x} = \frac{1}{2}$$
$$\Longrightarrow y = 2x^2 + x$$

which is shown in red, etc. *It's worth noting that each of the level curves is tangent to the line $y = x$...which just happens to be the boundary of the domain of f.*

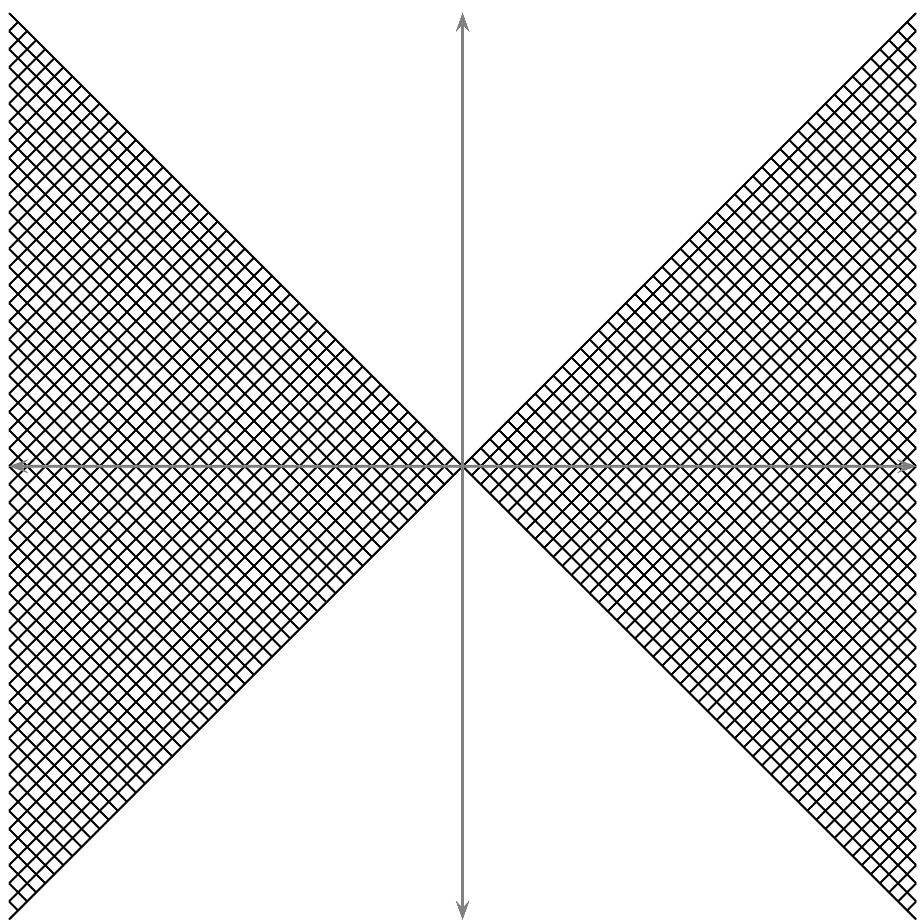

z
y
x

Example 3. More Level Curves

Let $g(x,\, y) = \sqrt{x^2 - y^2}$ and answer the same questions as above.

a. Find the domain of g.

Notice that we need to solve the inequality $y^2 \leq x^2$. It follows that $-|x| \leq y \leq |x|$ (why?). The *crosshatched* area below shows the domain.



b. Is the domain an open or closed region?

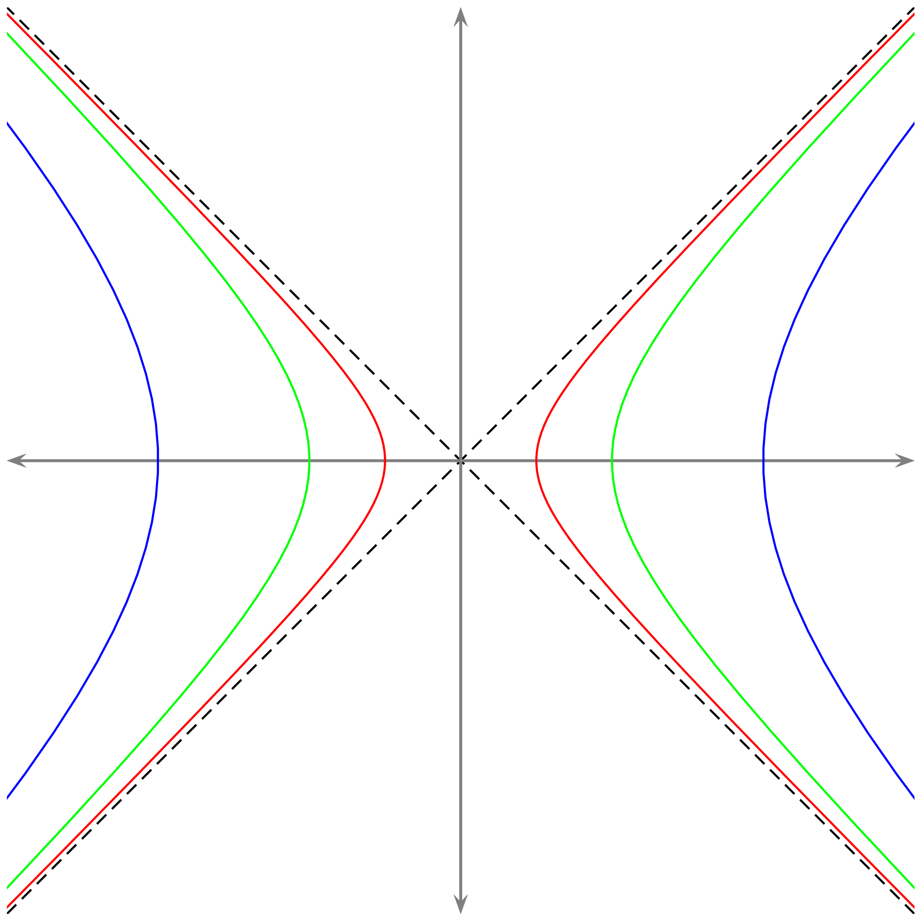
c. Is the domain bounded or unbounded?

d. Sketch the level curves.

To find these we should sketch the curves $g(x, y) = c$. We'll sketch a few for $c = 0, 1, 2,$ and 4. For example, if $c = 1$ we have

$$1 = \sqrt{x^2 - y^2} \Longrightarrow$$
$$1^2 = x^2 - y^2$$

It follows that each of the level curves is a hyperbola, which opens left-right (why?). For example, the level-curve $g(x, y) = 1$ is shown in red. Which of the "curves" corresponds to $c = 0$?



See the text for the three-dimensional analogues of the examples above.