12.6 Cylinders and Quadric Surfaces

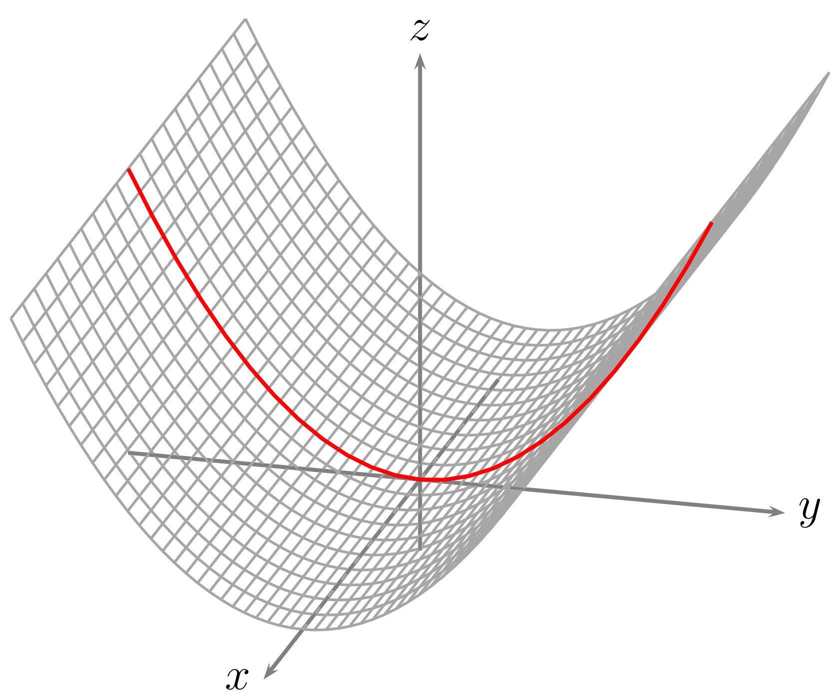
Cylinders

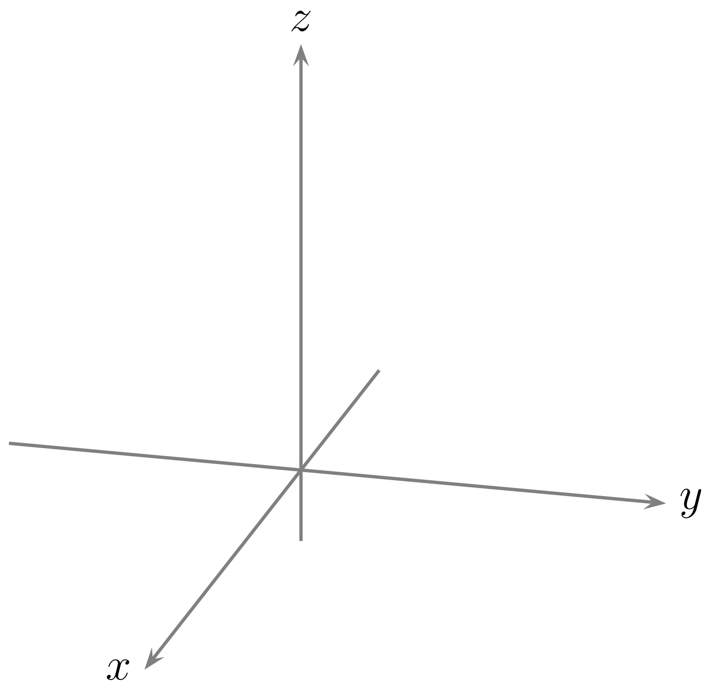
Definition. A **cylinder** is a surface that is generated by moving a straight line along a given planar curve while holding the line parallel to a fixed line. The curve is called a **generating curve**.

Remark. If the generating curve is a *circle* then the *cylinder* is a so-called **circular cylinder** from classical geometry. *Note: the generating curve is not unique.*

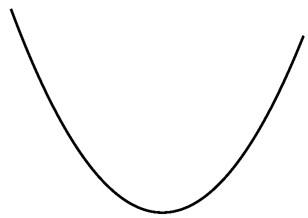
Example 1. The Parabolic Cylinder $z = y^2$

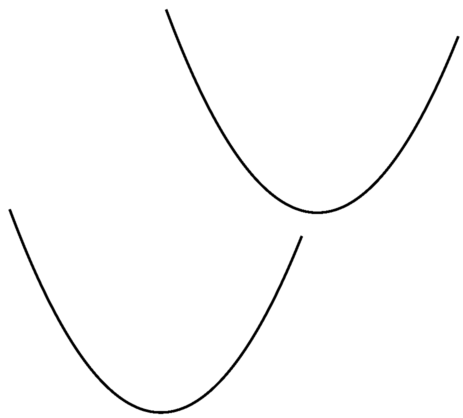
Since the x-coordinate is missing we guess that the generating curve lies in the yz-plane.

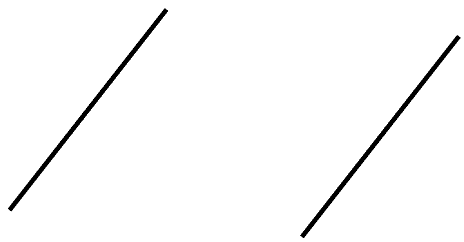


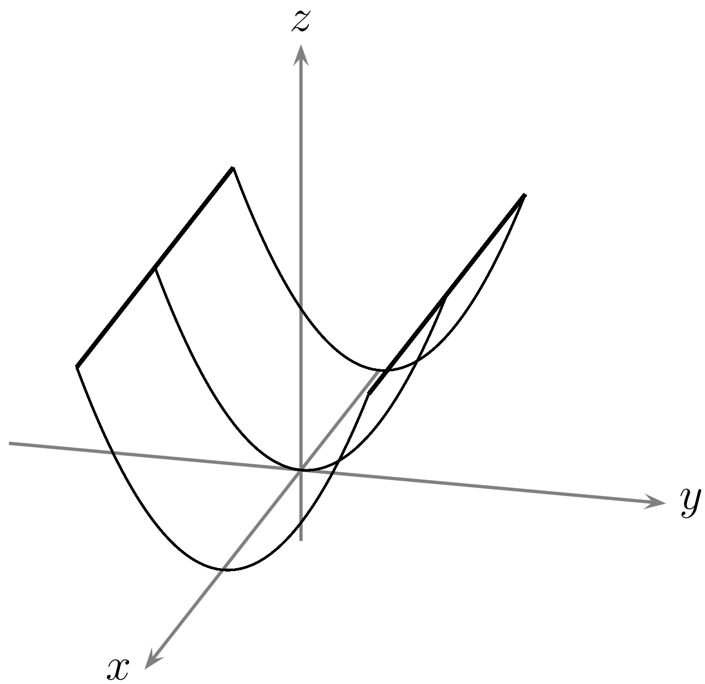


Here is a suggested method for sketching by hand.









z
y
x

Quadric Surfaces

Definition. A **quadric surface** is the graph in space of a second-degree equation in $x, y,$ and z. The most general form is

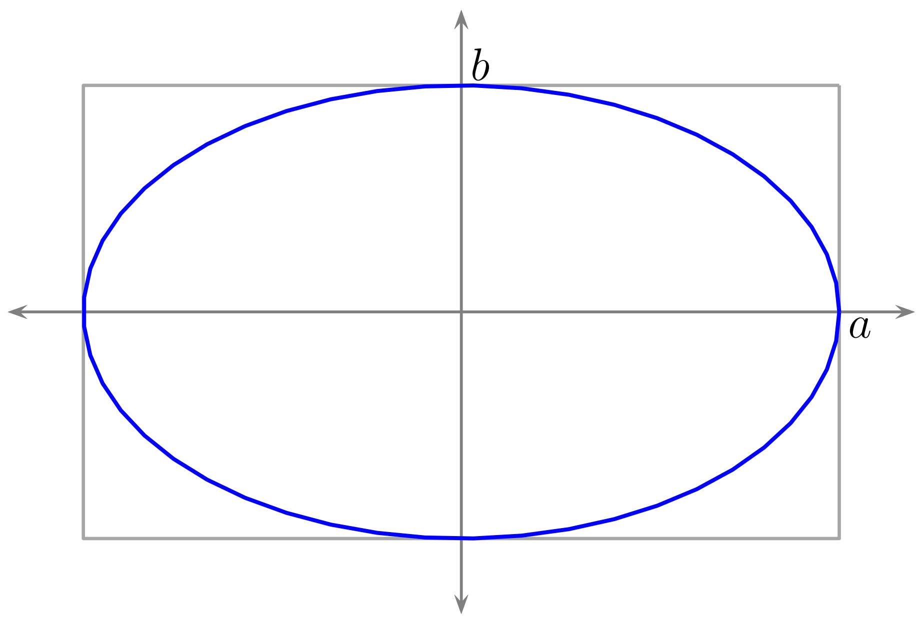
$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Jz + K = 0$$

where $A, B, C, \ldots$ are constants.

We’ve already seen two examples of quadric surfaces! We discuss several others.

Recall that the graph of the equation below is called an *ellipse* (centered at the origin). The graph intersects the coordinate axes at $(\pm a, 0)$ and $(0, \pm b)$.

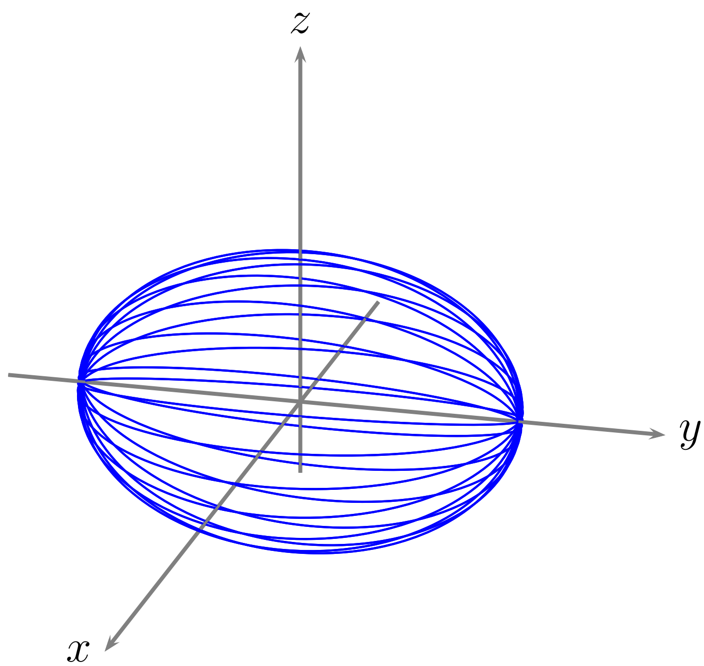
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



The three dimensional analogue of the example above is called an *ellipsoid*.

Example 2. Ellipsoids

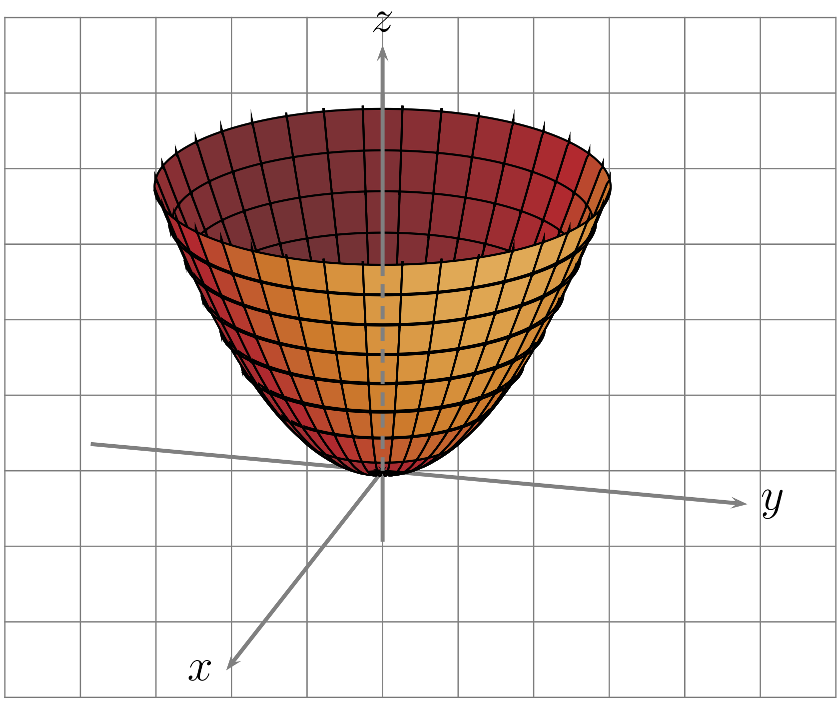
Sketch the graph of the equation below.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Paraboloids

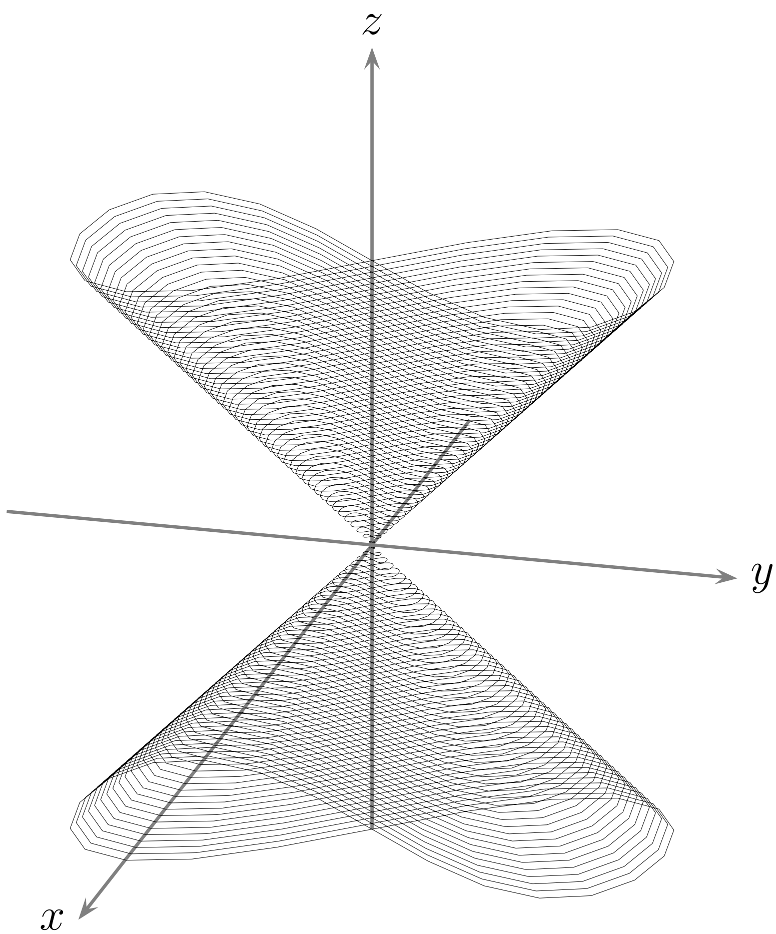
Example 3. A Circular Paraboloid



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

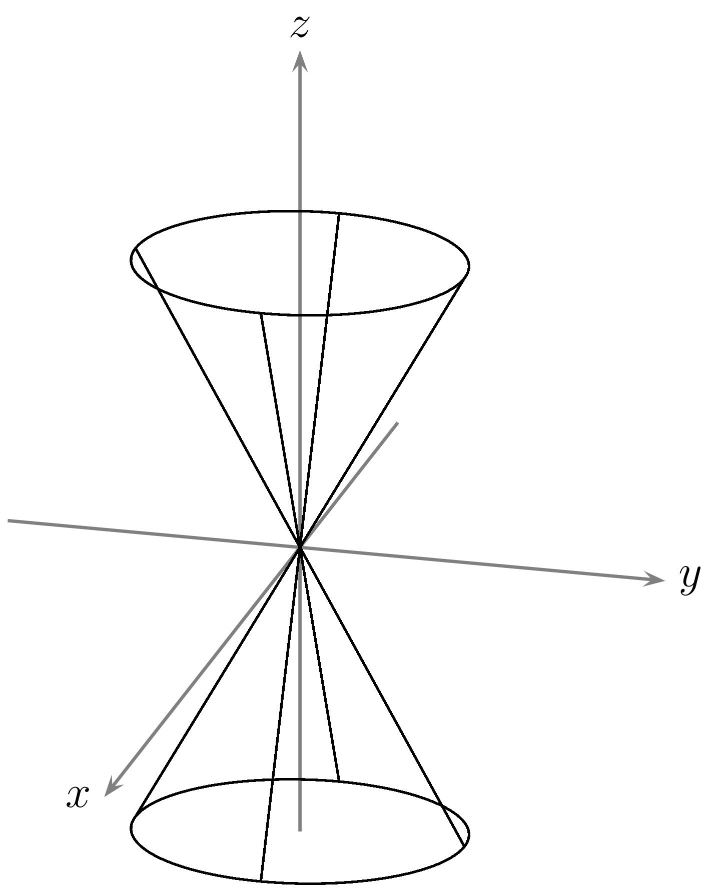
Cones

Example 4. Elliptical Cone

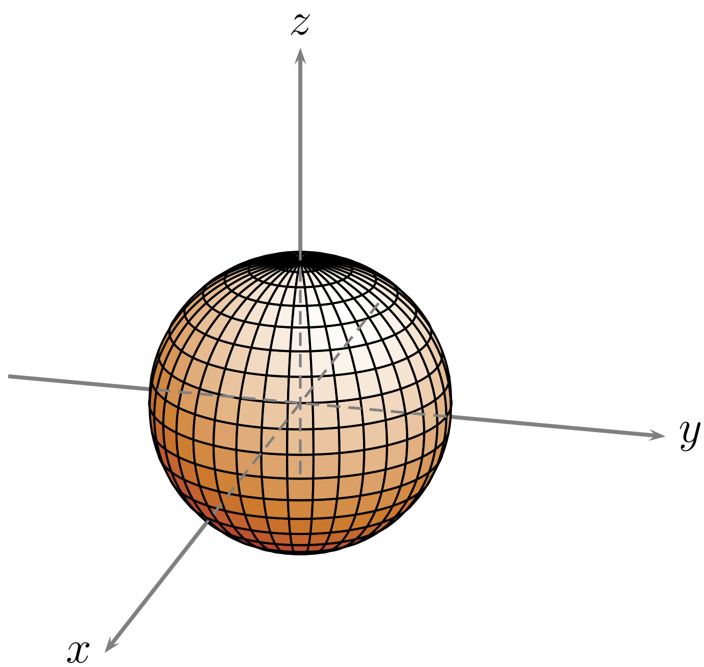


The general equation for these surfaces is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$


z
y
x

Spheres



As we saw in section 12.1, the equation of a sphere with radius r centered at the origin is

$$x^2 + y^2 + z^2 = r^2$$

Hyperboloids

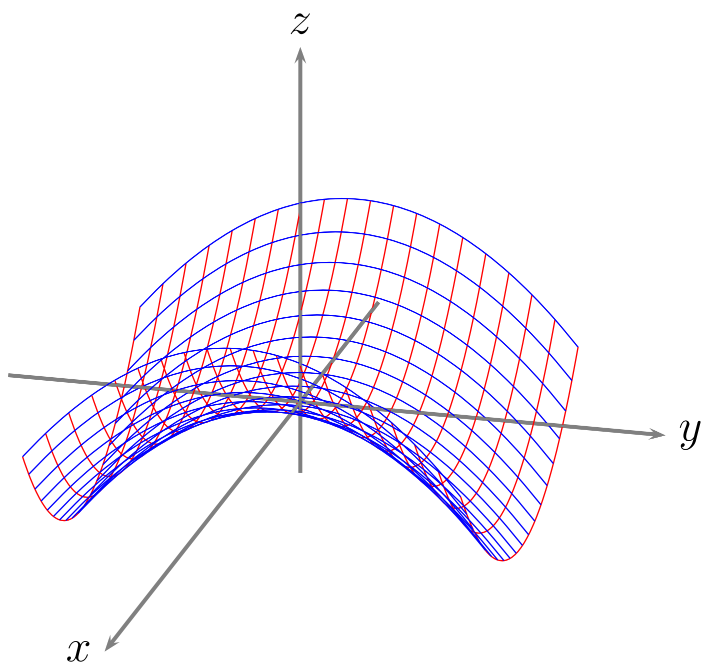
A **hyperbolic paraboloid** is given by the equation

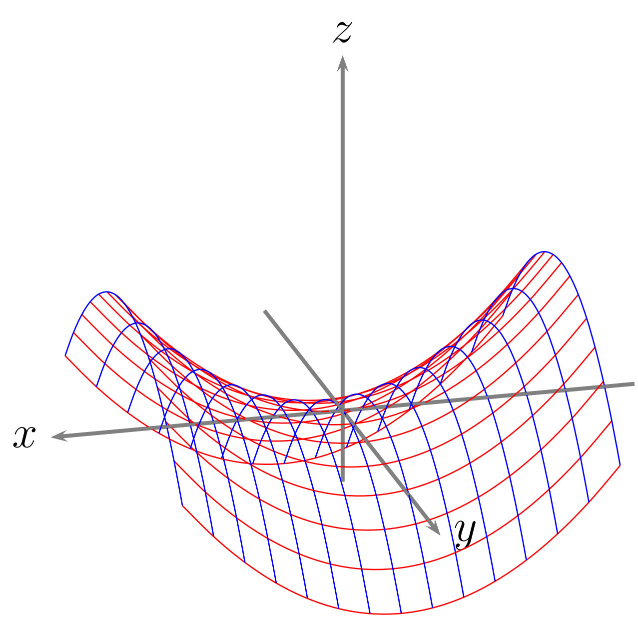
(1) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}, \quad c > 0$$

A prominent feature of these objects is a so-called *saddle point* which we will discuss in greater detail in chapter 14.

Example 5. A Saddle Point

The following two sketches show different views of the graph of equation (1).



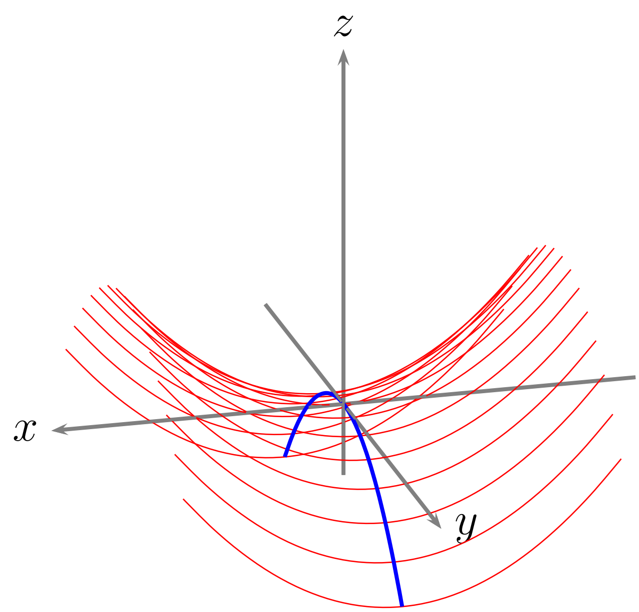


Now we consider planes parallel to each of the standard planes. In all cases, let k be a real constant.

1. The planes $y = k$. Then equation (1) yields.

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{k^2}{b^2}$$
$$\Longrightarrow z = A\,x^2 - B$$

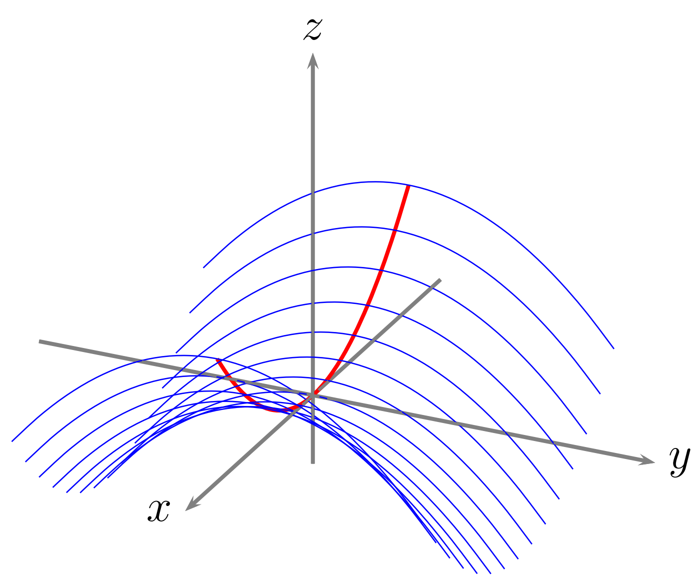
where A is a positive constant and B is a nonnegative constant (since k could be 0). Notice how the parabolas (shown in red above and below) straddle the parabola $z = \dfrac{-c}{b^2}\,y^2$ (shown in blue).



2. Now consider planes parallel to the yz-plane, that is, $x = k$. Then equation (1) yields.

$$\frac{z}{c} = -\frac{y^2}{b^2} + \frac{k^2}{a^2}$$
$$\Longrightarrow z = -C\,y^2 + D$$

where C is a positive constant and D is a nonnegative constant (again, since k could be 0). Notice how the parabolas (shown in blue above and below) straddle the parabola $z = \dfrac{c}{a^2}\,x^2$ (shown in red).

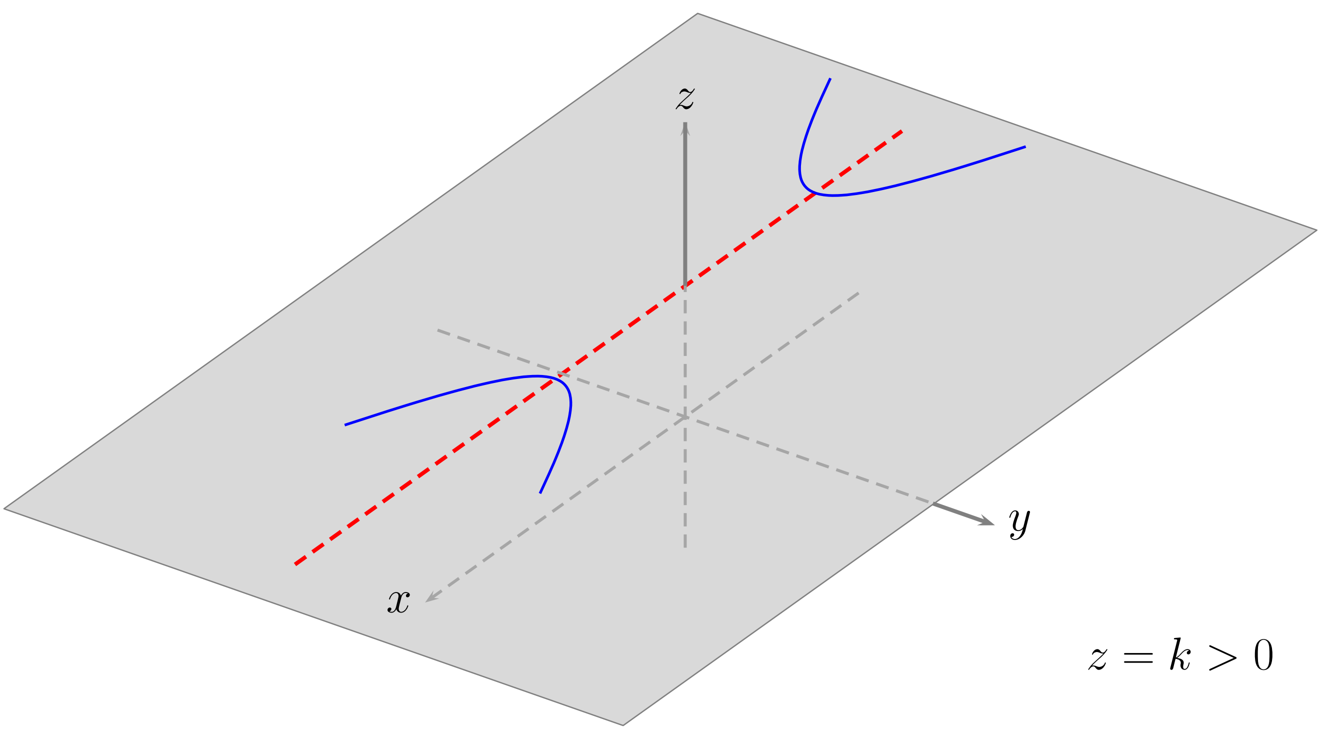


3. Finally, consider planes parallel to the xy-plane. That is, $z = k$. Then equation (1) yields.

$$\frac{k}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$
$$\Longrightarrow K = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

where K is a nonnegative constant. If $K = 0$ then we generate the big X as shown in class. Otherwise, we get hyperbolas as shown below.

For example, the intersection of the hyperbolic paraboloid with the plane $z = k > 0$ yields hyperbolas that open along the x-axis.



Example 6. Hyperboloids

The **hyperboloid of one sheet** is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

See the text.