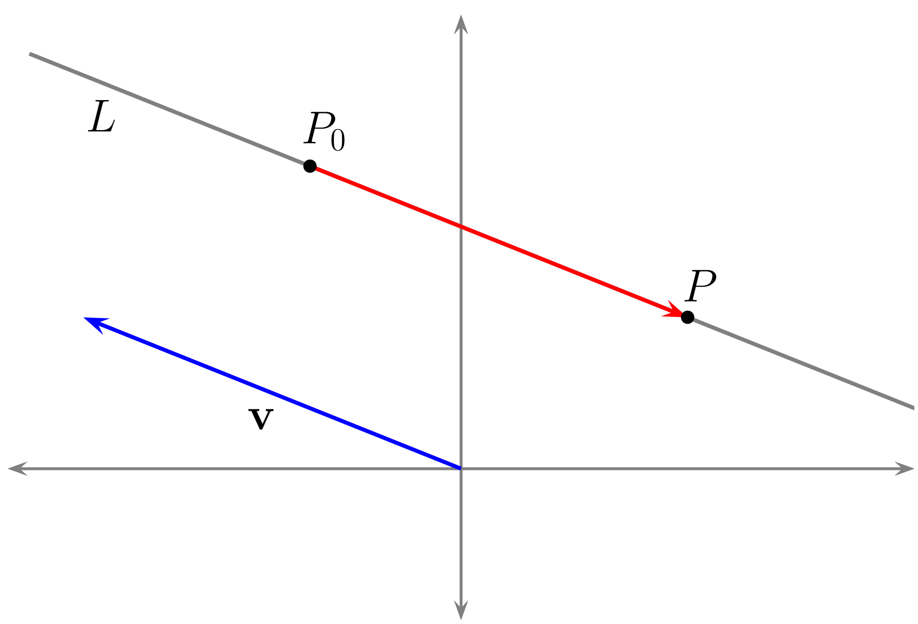
12.5 Lines and Planes in Space

We know from elementary geometry that (at least in the plane and in space) that two distinct points completely determine a line. In the plane, a line is also completely determined by one point and a direction (also called the *slope*). In this class we can use (unit) vectors to specify a direction.

The Vector Equation of a Line

Now consider the following problem. Find an equation of the line passing through the point $P_0 = P_0\,(x_0, y_0, z_0)$ in the direction of the vector $\mathbf{v}$.



Now if P is any other point on the line, L, then $\overline{P_0 P}$ must be parallel to the vector $\mathbf{v}$. In other words,

(1) $$\overline{P_0 P} = t\,\mathbf{v}, \quad t \in \mathbb{R}$$

Remark. Are there any other methods for determining if two nonzero vectors are parallel?

Now suppose that $P_0 = P_0\,(x_0, y_0, z_0)$ and $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$. Then equation (1) implies that $P = P\,(x, y, z)$ lies on the line if

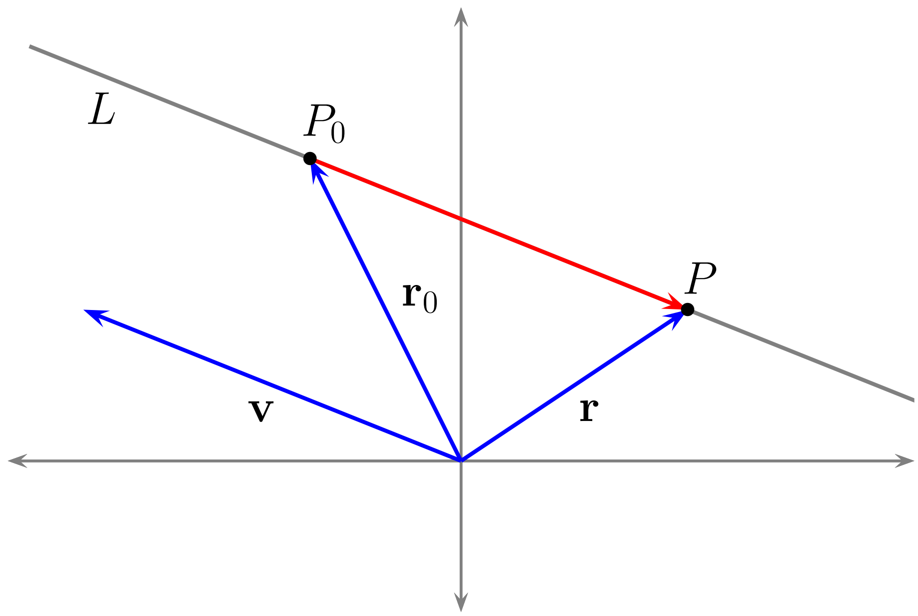
(2) $$(x - x_0)\,\mathbf{i} + (y - y_0)\,\mathbf{j} + (z - z_0)\,\mathbf{k} = t\,(v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k})$$

or

(3) $$x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} = x_0\,\mathbf{i} + y_0\,\mathbf{j} + z_0\,\mathbf{k} + t\,(v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k})$$

Now let $\mathbf{r}(t)$ be the position vector of a point $P(x, y, z)$ on the line (see sketch) and $\mathbf{r}_0$ be the position vector of the point P_0, then the vector equation of the line L through P_0 parallel to $\mathbf{v}$ is given by

(4) $$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad -\infty < t < \infty$$


L
P_0
$\mathbf{r}_0$
P
$\mathbf{v}$
$\mathbf{r}$

Recall that two vectors are equal if they have the same components. Now we can rewrite (4) in a more useful form.

Definition. Parametric Equations for a Line

The standard parametrization of the line through $P_0\,(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$ is

(5) $$x = x_0 + v_1 t,\ y = y_0 + v_2 t,\ z = z_0 + v_3 t,\ -\infty < t < \infty$$

Example 1. Find the parametric equations of the line through the points $P(-2, 1, 3)$ and $Q(-1, 4, 5)$. Here we are not given the vector $\mathbf{v}$ but one can easily be found. If we let $P_0 = Q$ and $\mathbf{v} = \overline{PQ}$ then

$$\begin{aligned}\mathbf{v} &= (-1 - (-2))\mathbf{i} + (4 - 1)\mathbf{j} + (5 - 3)\mathbf{k} \\ &= \mathbf{i} + 3\,\mathbf{j} + 2\,\mathbf{k}\end{aligned}$$

and equation 5 implies

$$x = -1 + t, \quad y = 4 + 3t, \quad z = 5 + 2t$$

Remark. Although choosing $P_0 = P$ results in a different parametrization it still produces the same line. *How can you verify this?*

The following technique will be very useful in some of the later sections.

Example 2. Find the parametric equations for the **line segment** joining the points $P(1,-1,4)$ and $Q(-3,2-3)$ (c.f., page 843 of the text).

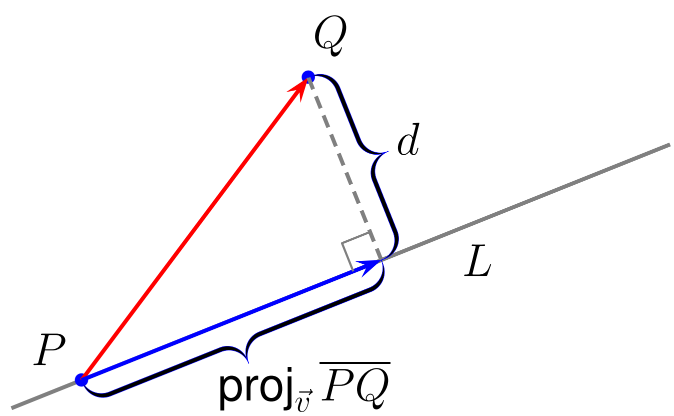
Let a and b be real numbers. Notice that for $0 \leq t \leq 1$ the function $f(t) = at + (1-t)b$ gives all real numbers between a and b and $f(0) = b$ and $f(1) = a$. Applying this idea for the $x, y,$ and z coordinates, we find that the line segment from P to Q is given by

$$x = -3t + (1-t)1$$
$$y = 2t + (1-t)(-1)$$
$$z = -3t + (1-t)4, \quad 0 \leq t \leq 1$$

Notice that this parametrization gives P when $t = 0$ and gives Q when $t = 1$. Notice that this method is quick and completely consistent with equation 5. *What happens if we drop the restriction on t?*

The Distance from a Point to a Line in Space

There are at least four different methods for finding this distance from a point to a line (including the one given by the text). We'll use the one suggested by the sketch below.



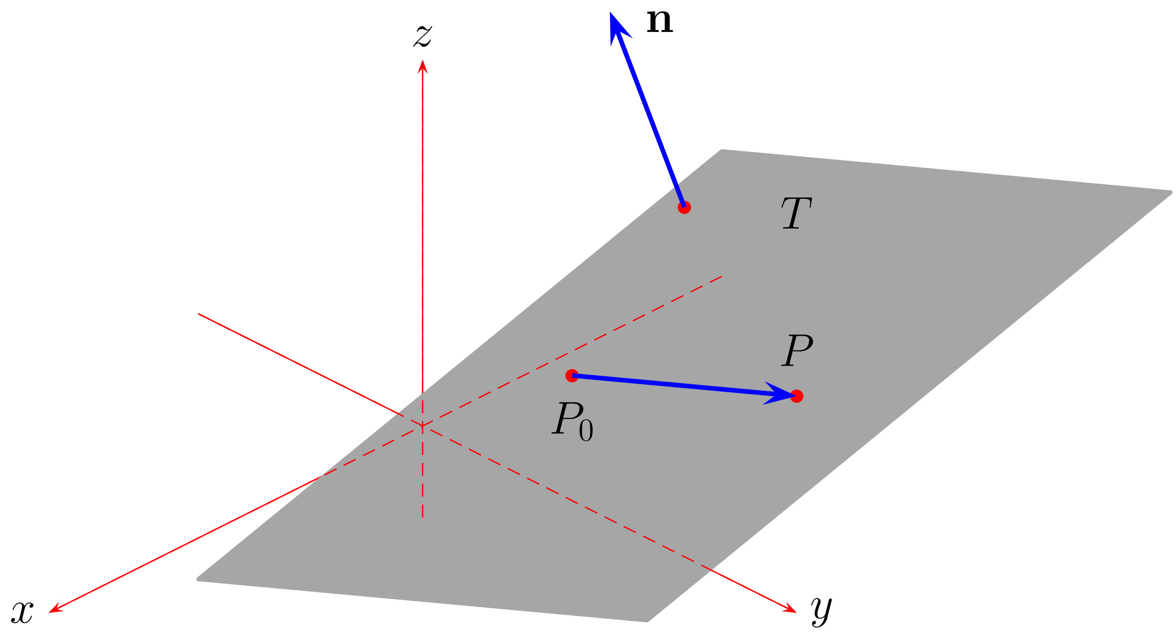
Let L be a line and Q be a point not on L. From elementary geometry, the distance d from Q to L is shown in the sketch. Now by the Pythagorean Theorem

$$d^2 + \|\text{proj}_{\vec{v}}\,\overline{PQ}\|^2 = \|\overline{PQ}\|^2$$

or

$$d = \sqrt{\|\overline{PQ}\|^2 - \|\text{proj}_{\vec{v}}\,\overline{PQ}\|^2}$$

The Equation of a Plane in Space



Let T be a plane in space and let $P_0 = P_0\left(x_0, y_0, z_0\right) \in T$. Also let $\mathbf{n} = A\,\mathbf{i} + B\,\mathbf{j} + C\,\mathbf{k}$ be a vector normal to T. If $P = P\left(x, y, z\right) \in T$, then $\mathbf{n} \cdot \overline{P_0 P} = 0$. In other words,

$$\{A\,\mathbf{i} + B\,\mathbf{j} + C\,\mathbf{k}\} \cdot \{(x - x_0)\,\mathbf{i} + (y - y_0)\,\mathbf{j} + (z - z_0)\,\mathbf{k}\} = 0$$

or

$$A\left(x - x_0\right) + B\left(y - y_0\right) + C\left(z - z_0\right) = 0$$

It follows that the equation of a plane containing $P_0 = P_0\left(x_0, y_0, z_0\right)$ normal to $\mathbf{n} = A\,\mathbf{i} + B\,\mathbf{j} + C\,\mathbf{k}$ is given by

(6) $$A\left(x - x_0\right) + B\left(y - y_0\right) + C\left(z - z_0\right) = 0$$

Remark. Equation 6 is often rewritten as

$$Ax + By + Cz = D$$

where $D = Ax_0 + By_0 + Cz_0$.

Lines of Intersection

If two nonparallel planes intersect the there intersection is a line. There is an easy way to find the (parametric) equations of this line. We proceed with an example.

Example 3. The Line of Intersection of Two Planes

Find the parametric equations for the line in which the planes $T_1\colon 2x - 4y + 3z = 12$ and $T_2\colon 3x + 3y - 2z = 6$ intersect.

There are some fancy techniques from calculus 3 that we might try, but a more straight-forward approach is to simply solve the (2×3) system. Let's *eliminate* z.

$$2T_1\colon 4x - 8y + 6z = 24$$

$$3T_2\colon 9x + 9y - 6z = 18$$

Adding left and right-hand sides respectively yields

$$13x + y = 42$$

Now, let $x = t$. Then

$$y = 42 - 13x = 42 - 13t$$

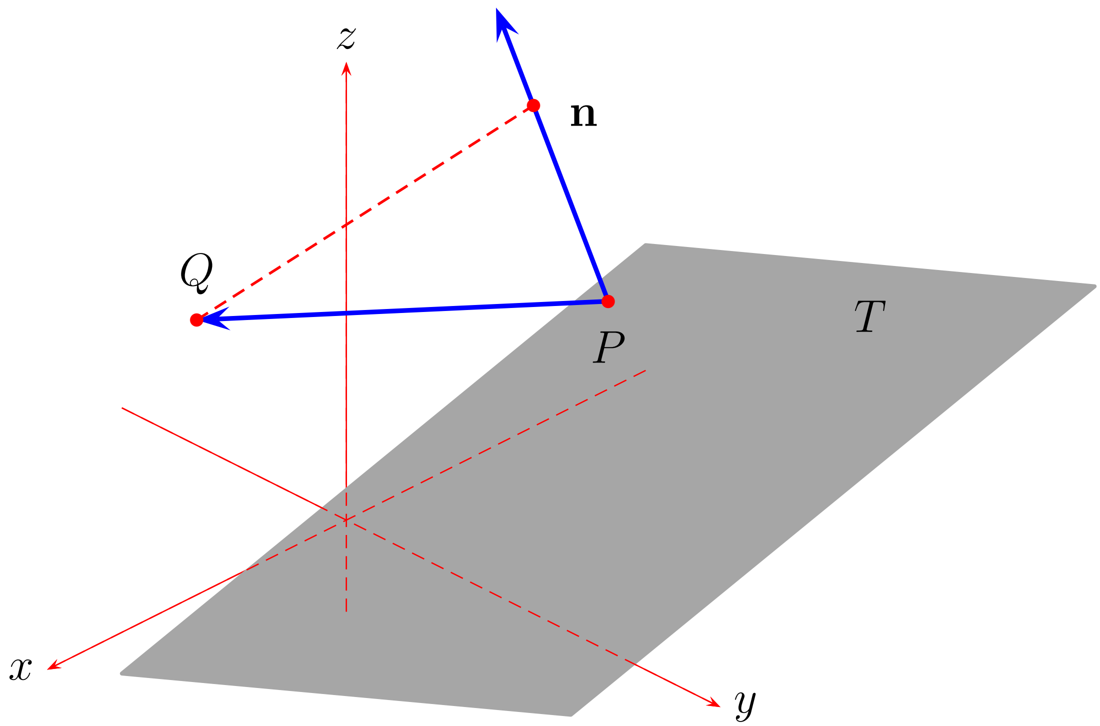
Finally, using either T_1 or T_2, we find z in terms of t to obtain

$$z = 60 - 18t$$

It follows that the parametric equations for the line of intersection are given by

$$x = t,\ y = 42 - 13t,\ z = 60 - 18t,\ -\infty < t < \infty$$

The Distance from a Point to a Plane



Example 4. Let T be the plane $2x + 5y + z = 2$ and $Q = Q(1, -1, 3)$. Notice that $Q \notin T$. Find the distance between the point Q and the plane T.

The sketch is very suggestive. If we can find a point P in the plane T, then it appears that the distance from Q to T is the magnitude of the projection of $\overrightarrow{PQ}$ onto $\mathbf{n}$. Let $P = P(-1, 0, 4)$. It is easy to see that $P \in T$. Now let $\mathbf{u} = \overrightarrow{PQ} = 2\,\mathbf{i} - \mathbf{j} - \mathbf{k}$. Since $\mathbf{n} = 2\,\mathbf{i} + 5\,\mathbf{j} + \mathbf{k}$, we have

$$\begin{aligned} \|\operatorname{proj}_{\mathbf{n}} \mathbf{u}\| &= \frac{|\mathbf{u} \cdot \mathbf{n}|}{\|\mathbf{n}\|} \\ &= \frac{|4 - 5 - 1|}{\sqrt{2^2 + 5^2 + 1^2}} = \frac{2}{\sqrt{30}} \end{aligned}$$

Angle Between Planes

See the text. *Note: By definition, the angle between two planes is acute, hence the formula given in the text (example 7) is wrong. It should be...*

(7) $$\theta = \cos^{-1}\left(\frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}\right)$$

Example 5. Find the angle between the planes from Example 3.

Let $\mathbf{n}_1 = 2\,\mathbf{i} - 4\,\mathbf{j} + 3\,\mathbf{k}$ and $\mathbf{n}_2 = 3\,\mathbf{i} + 3\,\mathbf{j} - 2\,\mathbf{k}$ be the normal vectors the planes T_1 and T_2, respectively. Then the cosine of the angle between planes T_1 and T_2 is

$$\begin{aligned}\cos\theta &= \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} \\ &= \frac{|-12|}{\sqrt{29}\,\sqrt{22}} \\ &= \frac{12}{\sqrt{29}\sqrt{22}}\end{aligned}$$

It follows that

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{12}{\sqrt{29}\sqrt{22}}\right) \\ &\approx 1.075736212\end{aligned}$$