12.4 Cross Products

Suppose that $\mathbf{A}$ and $\mathbf{B}$ are non-zero and non-parallel vectors in $\mathbb{R}^3$. Then the vectors determine a plane. Let θ be the angle from $\mathbf{A}$ to $\mathbf{B}$. Let $\mathbf{n}$ be a unit vector whose direction is orthogonal to the plane determined by $\mathbf{A}$ and $\mathbf{B}$ according to the **right-hand** rule. We define a new vector, $\mathbf{A} \times \mathbf{B}$ ($\mathbf{A}$ cross $\mathbf{B}$) to be a vector in the direction of $\mathbf{n}$ and whose magnitude is given by

(1) $$\mathbf{A} \times \mathbf{B} = (\|\mathbf{A}\| \, \|\mathbf{B}\| \, \sin\theta)\, \mathbf{n}$$

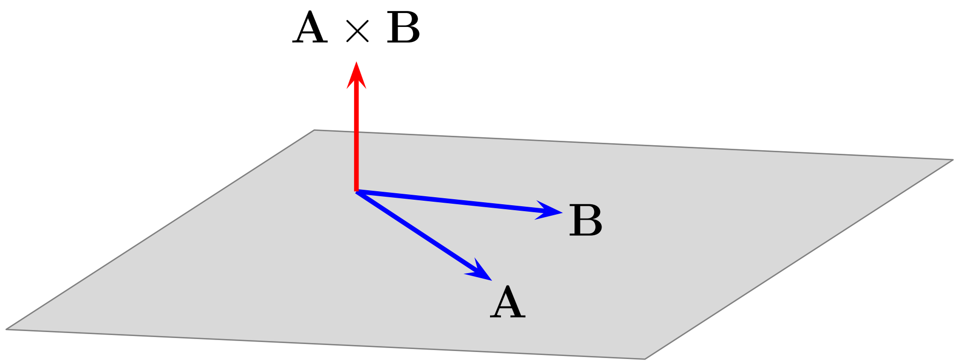
Notice that we have the following equivalence.

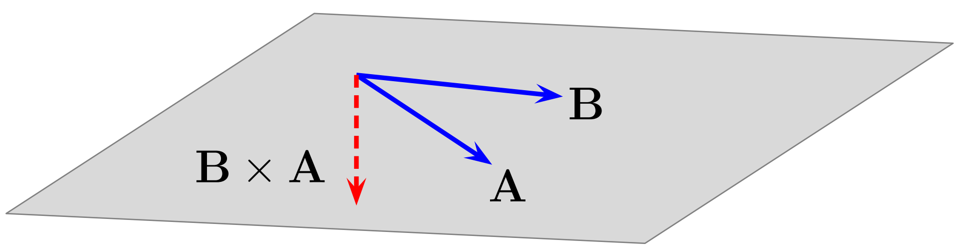
(2) Nonzero vectors $\mathbf{A}$ and $\mathbf{B}$ are parallel $\iff$ $\mathbf{A} \times \mathbf{B} = \mathbf{0}$

and that

(3) $$\mathbf{B} \times \mathbf{A} = -(\mathbf{A} \times \mathbf{B})$$

(See the sketch below.)


$\mathbf{A} \times \mathbf{B}$
$\mathbf{B}$
$\mathbf{A}$


$\mathbf{B}$
$\mathbf{B} \times \mathbf{A}$
$\mathbf{A}$

Example 1. Cross products with the standard unit vectors.

(4) $$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

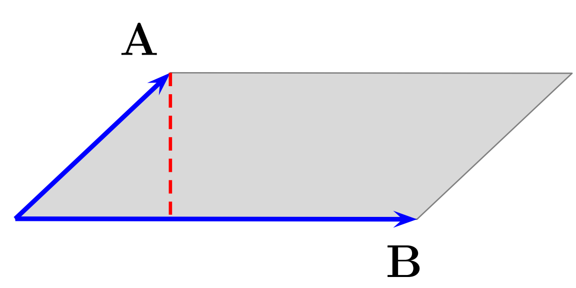
(5) $$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$

(6) $$\mathbf{k} \times \mathbf{i} = \mathbf{j}$$

Since $\mathbf{n}$ is a unit vector, the magnitude of $\mathbf{A} \times \mathbf{B}$ is

$$\begin{aligned} \|\mathbf{A} \times \mathbf{B}\| &= \|\mathbf{A}\|\,\|\mathbf{B}\|\,|\sin\theta|\,\|\mathbf{n}\| \\ &= \|\mathbf{A}\|\,\|\mathbf{B}\|\,\sin\theta \quad (\text{since } \sin\theta \geq 0) \end{aligned}$$

which is the area of the parallelogram in the sketch below.



Algebraic Properties of the Cross Product

Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be vectors and x, y be scalars. Then

Scalar Distributive Law

$$(x\mathbf{A}) \times (y\mathbf{B}) = (xy)(\mathbf{A} \times \mathbf{B})$$

Vector Distributive Laws

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$$

$$(\mathbf{B} + \mathbf{C}) \times \mathbf{A} = (\mathbf{B} \times \mathbf{A}) + (\mathbf{C} \times \mathbf{A})$$

Miscellaneous Laws

$$\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A})$$

$$\mathbf{0} \times \mathbf{A} = \mathbf{0}$$

Remark. It is worth noting that the cross product is NOT associative. That is,

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \neq \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$$

Is the dot product associative?

Determinant Formula for Cross Products

There is a nice way to compute the cross product based on equation 3, example 1 and the distributive laws described above.

Recall that the *determinant* of a 2×2 matrix is given by the following formula.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Now let $\mathbf{A} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ and $\mathbf{B} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$. Then

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= (a_2\,b_3 - a_3\,b_2)\,\mathbf{i} - (a_1\,b_3 - a_3\,b_1)\,\mathbf{j} + (a_1\,b_2 - a_2\,b_1)\,\mathbf{k} \end{aligned}$$

Example 2. Computing a Vector Cross Product

Let $\mathbf{A} = \mathbf{i} + 3\,\mathbf{j} - 2\,\mathbf{k}$ and $\mathbf{B} = 3\,\mathbf{i} + 2\,\mathbf{k}$.

Find $\mathbf{A} \times \mathbf{B}$.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ 3 & 0 & 2 \end{vmatrix}$$

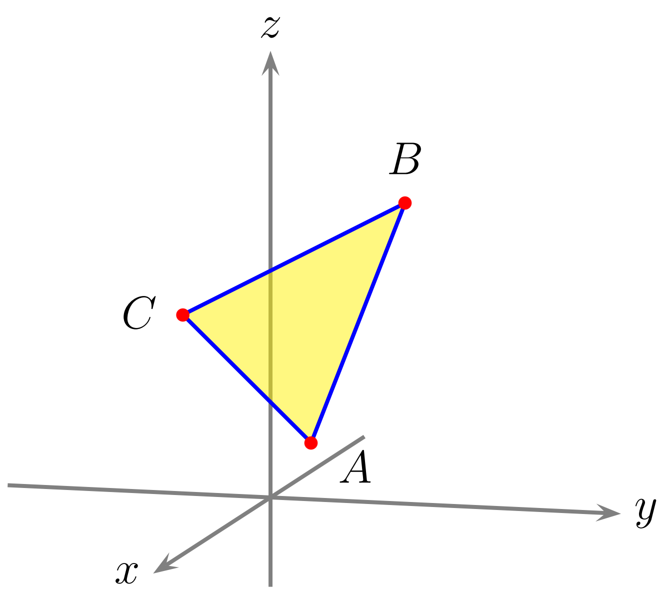
$$= \mathbf{i} \begin{vmatrix} 3 & -2 \\ 0 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 3 \\ 3 & 0 \end{vmatrix}$$

$$= (6 - 0)\,\mathbf{i} - (2 - (-6))\,\mathbf{j} + (0 - 9)\,\mathbf{k}$$

$$= 6\,\mathbf{i} - 8\,\mathbf{j} - 9\,\mathbf{k}$$

Recall that three non-collinear points determine a plane. The following example illustrates how we can use cross products to find vectors that are normal (orthogonal) to this plane.

Example 3. Consider the points $A(2,1,1), B(-2,1,3), C(0,-1,2)$. Find the area of $\triangle ABC$ and find a unit vector normal to the plane ABC.



Let

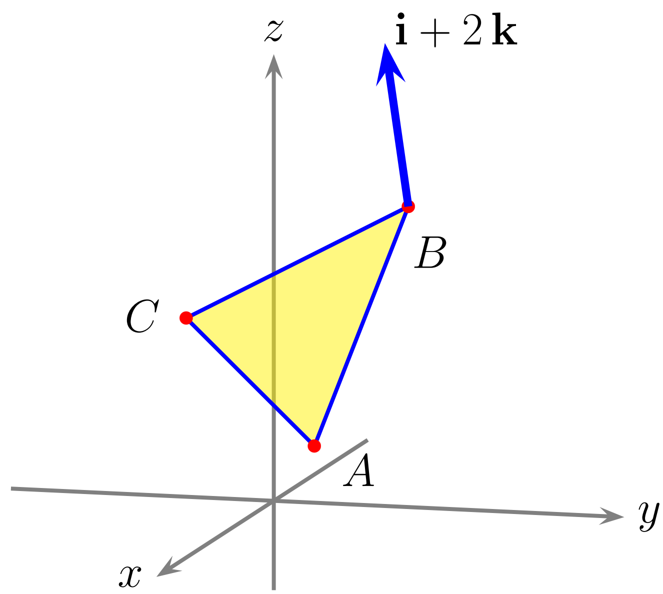
$$\mathbf{u} = \overline{AB} = -4\,\mathbf{i} + 2\,\mathbf{k}$$
$$\mathbf{v} = \overline{AC} = -2\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$$

Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 0 & 2 \\ -2 & -2 & 1 \end{vmatrix}$$

$$= \mathbf{i} \begin{vmatrix} 0 & 2 \\ -2 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -4 & 2 \\ -2 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -4 & 0 \\ -2 & -2 \end{vmatrix}$$

$$= 4\,\mathbf{i} + 8\,\mathbf{k}$$



It follows that $\mathbf{n} = \dfrac{4}{\sqrt{80}}\,(\,\mathbf{i} + 2\,\mathbf{k})$ is a unit vector normal to the given plane. *Can you find another?*

Explain why the area of $\triangle ABC$ is $\sqrt{80}/2$.