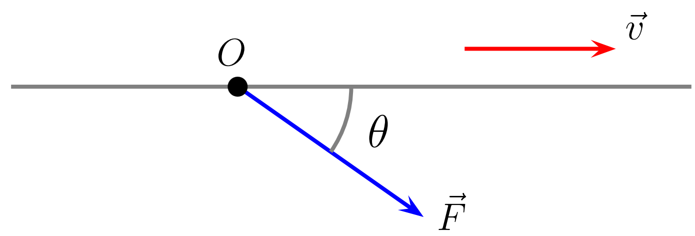
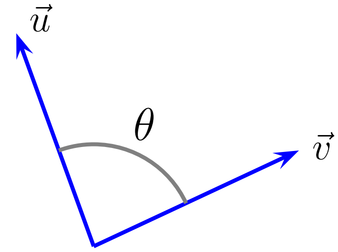
12.3 The Dot Product

Suppose that a force is applied to an object as shown below.



Now suppose we want to find the magnitude of the force $\vec{F}$ in the direction of $\vec{v}$ (horizontal in this case). Then we need to find the angle θ.

To do this we suppose that two nonzero vectors $\vec{u}$ and $\vec{v}$ are placed so that their initial points coincide. Then they form an angle θ (in the plane containing the two vectors) with $0 \leq \theta \leq \pi$.



Now $\theta = 0$ if $\vec{u}$ and $\vec{v}$ are pointed in the same direction and $\theta = \pi$ if they are pointed in the opposite direction. In general we have,

Theorem 1. Angle between two vectors. The angle θ between two nonzero vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is given by

(1)
$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{\|\vec{u}\|\|\vec{v}\|}\right)$$

The proof is a consequence of the Law of Cosines (see the text). The numerator of the right-hand quantity in equation (1) is important enough to have a name.

Definition. Dot Product

The dot product of two vectors $\vec{u}$ and $\vec{v}$ is

(2)
$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

Remark. The dot product is also called the inner or scalar product. *Notice that the dot product of two vectors is a scalar.*

Example 1.

Let $\vec{u} = \langle 1, 2, -3 \rangle$ and $\vec{v} = \langle -2, 3, 2 \rangle$. Find $\vec{u} \cdot \vec{v}$ and the angle θ between these vectors. We have $\|\vec{u}\| = \sqrt{14}$ and $\|\vec{v}\| = \sqrt{17}$. Thus

$$\begin{aligned} \vec{u} \cdot \vec{v} &= 1(-2) + 2(3) + (-3)(2) \\ &= -2 \end{aligned}$$

and

$$\theta = \cos^{-1}\left(\frac{-2}{\sqrt{14}\sqrt{17}}\right)$$

In view of Theorem 1 we have the following alternate definition of the dot product.

(3) $$\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$.

Orthogonality

Two nonzero vectors are called **orthogonal** (or perpendicular) if the angle between them is $\pi/2$. Can we use the dot product to determine if two vectors are orthogonal?

Suppose that $\vec{u}$ and $\vec{v}$ are orthogonal. Then by (3) we know

$$\begin{aligned}\vec{u}\cdot\vec{v} &= \|\vec{u}\|\|\vec{v}\|\cos\pi/2\\ &= \|\vec{u}\|\|\vec{v}\|(0)\\ &= 0\end{aligned}$$

On the other hand, if $\vec{u}, \vec{v} \neq \mathbf{0}$ then

$$\begin{aligned}\vec{u}\cdot\vec{v} &= 0\\ \Longrightarrow \cos\theta &= 0\\ \Longrightarrow \theta &= \pi/2\end{aligned}$$

These observations lead to the following definition.

Definition. Orthogonal Vectors

Vectors $\vec{u}$ and $\vec{v}$ are **orthogonal** if and only if $\vec{u}\cdot\vec{v} = 0$.

Example 2.

a. Let $\vec{u} = 3\,\mathbf{i} + 4\,\mathbf{j}$ and $\vec{v} = 4/5\,\mathbf{i} - 3/5\,\mathbf{j}$. Show that $\vec{u}$ and $\vec{v}$ are orthogonal. So

$$\begin{aligned}\vec{u}\cdot\vec{v} &= (3\,\mathbf{i} + 4\,\mathbf{j})\cdot(4/5\,\mathbf{i} - 3/5\,\mathbf{j})\\ &= 12/5 - 12/5\\ &= 0\end{aligned}$$

b. Clearly $\mathbf{0}\cdot\vec{v} = 0$ hence $\mathbf{0}$ is orthogonal to every vector.

Proposition 2. Properties of the Dot Product

If $\vec{u}, \vec{v},$ and $\vec{w}$ are vectors and $k \in \mathbb{R}$ then

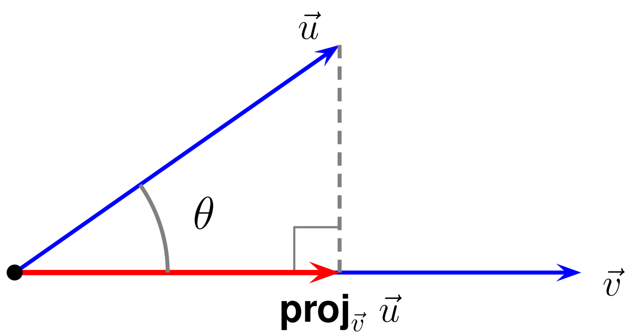
1. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
2. $(k\vec{u})\cdot\vec{v} = \vec{u}\cdot(k\vec{v}) = k\,(\vec{u}\cdot\vec{v})$
3. $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$
4. $\vec{u}\cdot\vec{u} = \|\vec{u}\|^2$
5. $\mathbf{0}\cdot\vec{u} = 0$

Proof. These are easy consequences of the definitions and are left as exercises.

Remark. Item 2 above implies that if $\vec{u}$ is orthogonal to $\vec{v}$ then $\vec{u}$ is orthogonal to every scalar multiple of $\vec{v}$.

Vector Projections

Consider the problem posed at the beginning of this section (a modified form is illustrated in the sketch below).



We need to find a formula for **proj**$_{\vec{v}}\ \vec{u}$. Since we already know the direction (why?), all that we need is the length of **proj**$_{\vec{v}}\ \vec{u}$. From trigonometry we have

$$\cos\theta = \frac{\|\,\mathbf{proj}_{\vec{v}}\ \vec{u}\|}{\|\vec{u}\|}$$

$$\Longrightarrow \|\,\mathbf{proj}_{\vec{v}}\ \vec{u}\| = \|\vec{u}\|\cos\theta = \|\vec{u}\|\cos\theta\,\frac{\|\vec{v}\|}{\|\vec{v}\|}$$

$$= \frac{\|\vec{u}\|\|\vec{v}\|\cos\theta}{\|\vec{v}\|}$$

$$= \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|} \quad \text{(a more convenient form)}$$

Recall that the direction of $\vec{v}$ is given by $\vec{v}/\|\vec{v}\|$. It follows that

$$\begin{aligned}\textbf{proj}_{\vec{v}}\ \vec{u} &= \|\ \textbf{proj}_{\vec{v}}\ \vec{u}\| \times \frac{\vec{v}}{\|\vec{v}\|} \\ &= \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}\right)\left(\frac{\vec{v}}{\|\vec{v}\|}\right) \\ &= \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v} \\ &= \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\ \vec{v}\end{aligned}$$

In other words, the vector projection of $\mathbf{u}$ onto $\vec{v}$ is given by

$$\textbf{proj}_{\vec{v}}\ \vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\ \vec{v} \tag{4}$$

Remark. Technically, there's a mistake in the logic above. What is it? The result is correct but the first line assumes that θ is acute. Technically, the first line should be

$$\textbf{proj}_{\vec{v}}\ \vec{u} = \pm\|\ \textbf{proj}_{\vec{v}}\ \vec{u}\| \times \frac{\vec{v}}{\|\vec{v}\|}$$

The sign of the RHS of the expression is positive if θ is acute and negative otherwise. However, this all comes out in the wash with the final formula above.

Example 3. Let $\vec{u} = 3\,\mathbf{i} + 2\,\mathbf{j}$. Find $\textbf{proj}_{\vec{v}}\ \vec{u}$ for each of the following vectors $\vec{v}$ below.

a. $\vec{v} = \ \mathbf{i}$

   This one is easy. It should be $\textbf{proj}_{\vec{v}}\ \vec{u} = 3\,\mathbf{i}$. Do you see why?

b. $\vec{v} = 2\,\mathbf{i}$

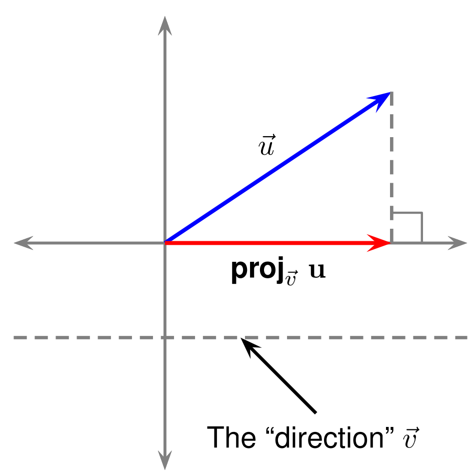
   Should also be $\textbf{proj}_{\vec{v}}\ \vec{u} = 3\,\mathbf{i}$.

c. $\vec{v} = -6\,\mathbf{i}$

   Should also be $\textbf{proj}_{\vec{v}}\ \vec{u} = 3\,\mathbf{i}$. Here's the calculation for this one. So by (4) we have

$$\begin{aligned}\textbf{proj}_{\vec{v}}\ \vec{u} &= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v} \\ &= \frac{3(-6)}{36}\,(-6)\,\mathbf{i} \\ &= 3\,\mathbf{i}\end{aligned}$$

In this example, the vector $\vec{v}$ is clearly in the $\pm\,\mathbf{i}$ direction in all 3 cases. As a consequence, the projection (shown in red in the sketch) must be the same in all 3 cases.



Example 4. Finding a vector projection.

Let $\vec{u} = -3\,\mathbf{i} + 2\,\mathbf{j}$ and $\vec{v} = 2\,\mathbf{i} + \mathbf{j}$. Find $\mathbf{proj}_{\vec{v}}\,\vec{u}$ and $\text{orth}_{\vec{v}}\,\vec{u}$.

Notice that

$$\vec{u} \cdot \vec{v} = (-3)(2) + 2(1) = -4$$
$$\vec{v} \cdot \vec{v} = 2^2 + 1^2 = 5$$

a. $\mathbf{proj}_{\vec{v}}\,\vec{u} = \frac{-4}{5}\,\vec{v} = \frac{-4}{5}\,(2\,\mathbf{i} + \mathbf{j})$.

b. We have $\text{comp}_{\vec{v}}\,\vec{u} =^{\text{def}} \|\mathbf{proj}_{\vec{v}}\,\vec{u}\| = \frac{4}{\sqrt{5}}$. Why?

Example 5. Vector Decomposition

Often, it is important to rewrite a vector as the sum of two other vectors (as strange as this may seem). Using the vectors from the previous example, rewrite $\vec{u}$ as the sum of two vectors, one parallel to $\vec{v}$ and the other orthogonal to $\vec{v}$.

From Example 4,

$$\begin{aligned} -3\,\mathbf{i} + 2\,\mathbf{j} &= \textbf{proj}_{\vec{v}}\ \vec{u} + \overbrace{(\vec{u} - \textbf{proj}_{\vec{v}}\ \vec{u})}^{\textbf{orth}_{\vec{v}}\ \vec{u}} \\ &= \frac{-4}{5}\,(2\,\mathbf{i} + \mathbf{j}) + \left((-3\,\mathbf{i} + 2\,\mathbf{j}) - \left(\frac{-4}{5}\,(2\,\mathbf{i} + \mathbf{j})\right)\right) \\ &= \frac{-4}{5}\,(2\,\mathbf{i} + \mathbf{j}) + \frac{7}{5}(-\mathbf{i} + 2\,\mathbf{j}) \end{aligned}$$