

Example 1. Let $f(x, y) = \frac{3y\sqrt{x}}{2}$ and let C be given by the vector equation

$$(1) \quad \mathbf{r}(t) = t \mathbf{i} + \frac{2t^{3/2}}{3} \mathbf{j}, \quad 0 \leq t \leq 3.$$

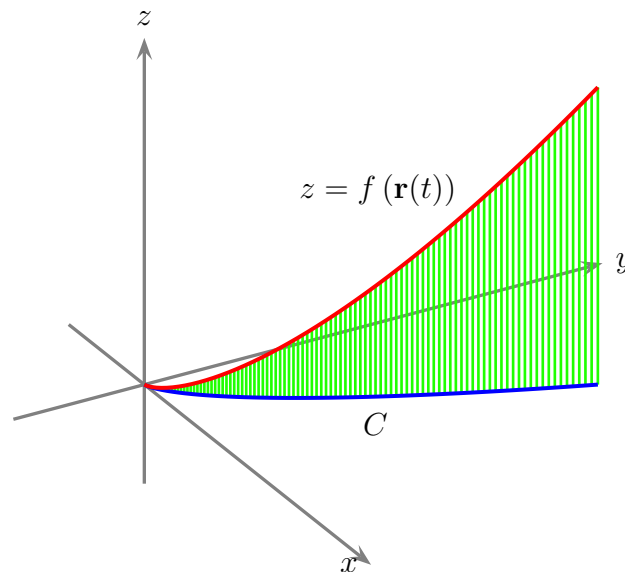
Evaluate the line integral

$$(2) \quad \int_C f(x, y) ds.$$

Using the substitution $u = 1 + t$, it is not difficult to show that

$$\begin{aligned} \int_C f(x, y) ds &= \int_0^3 f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt \\ &= \int_0^3 t^2 \sqrt{1+t} dt \\ &= \int_1^4 (u-1)^2 \sqrt{u} du \\ &= \vdots \\ &= \left(\frac{2u^{7/2}}{7} - \frac{4u^{5/2}}{5} + \frac{2u^{3/2}}{3} \right) \Big|_1^4 \\ &= \frac{1696}{105} \approx 16.15 \end{aligned}$$

But what does it mean? The sketch below shows the graph of the function $z = f(x, y)$ for $(x, y) \in C$.



A quite reasonable interpretation is that the integral in (2) is the “area under the curve” (shown in green). Our experience from first semester calculus suggests that we might approximate this integral using Riemann Sums. That is, we will use the estimate

$$(3) \quad \int_C f(x, y) ds \approx \sum_{j=1}^n f(x_j, y_j) \Delta s_j$$

for a suitably chosen partition of the arc length interval.

In section 13.3 we learned that the arc length parameter is given by

$$\begin{aligned} s(t) &= \int_0^t |\mathbf{r}'(\tau)| d\tau \\ &= \int_0^t \sqrt{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2} d\tau \\ &= \int_0^t \sqrt{1 + \tau} d\tau \end{aligned}$$

It follows that

$$(4) \quad s(t) = \frac{2}{3} ((1 + t)^{3/2} - 1)$$

Notice that $s(0) = 0$ and $s(3) = 14/3$. So we must partition the interval $S = [0, 14/3]$ and proceed in the usual manner. It is a little easier to partition the t -interval $[0, 3]$ and work with the induced partition on S . Using (1) and (4) we construct the table below. (E.g., $t_6 = 1.5$, $s_9 = 3.24$, etc.)

index	t	s	x	y	$f(x, y)$
0	0	0	0	0	0
1	0.25	0.27	0.25	0.08	0.06
2	0.50	0.56	0.50	0.24	0.25
3	0.75	0.88	0.75	0.43	0.56
4	1.00	1.22	1.00	0.67	1.00
5	1.25	1.58	1.25	0.93	1.56
6	1.50	1.97	1.50	1.22	2.25
7	1.75	2.37	1.75	1.54	3.06
8	2.00	2.80	2.00	1.89	4.00
9	2.25	3.24	2.25	2.25	5.06
10	2.50	3.70	2.50	2.64	6.25
11	2.75	4.17	2.75	3.04	7.56
12	3.00	4.67	3.00	3.46	9.00

Here we have chosen a partition of $[0, 3]$ with 12 equal subintervals.

Now let $\Delta s_j = s_j - s_{j-1}$, $j = 1, 2, \dots, 12$. Then the upper and lower Riemann Sums for the given interval are easy to calculate.

$$\begin{aligned}\text{Lower Riemann Sum} &= \sum_{j=1}^{12} f(s_{j-1}) \Delta s_j \\ &= 0 \cdot (0.27) + 0.06 \cdot (0.29) + \dots + 7.56 \cdot (0.5) \\ &\approx 13.69\end{aligned}$$

and

$$\begin{aligned}\text{Upper Riemann Sum} &= \sum_{j=1}^{12} f(s_j) \Delta s_j \\ &= 0.06 \cdot (0.27) + 0.25 \cdot (0.29) + \dots + 9.0 \cdot (0.5) \\ &\approx 18.12\end{aligned}$$

Notice that

$$13.69 \leq \int_C f(x, y) ds = \frac{1696}{105} \leq 18.12$$

as we might expect.

There is a bit more that we can learn from this example. By the Fundamental Theorem of Calculus (or a direct calculation),

$$s'(t) = \sqrt{1+t} > 0 \text{ for } t \in (0, 3)$$

Hence s is increasing on T and, as such, is invertible. It follows from (4) that

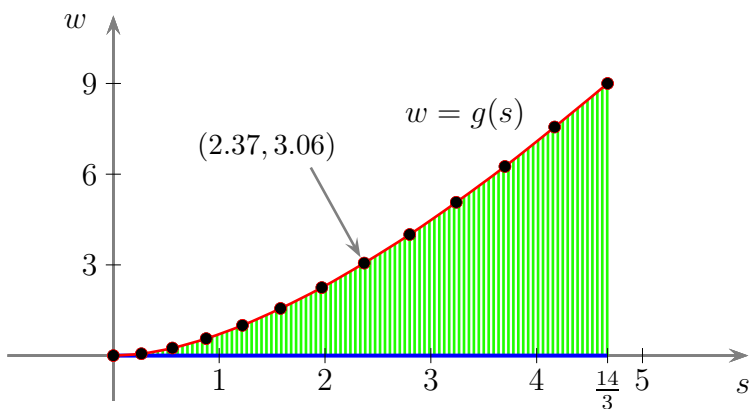
$$(5) \quad t = \left(\frac{3s}{2} + 1\right)^{2/3} - 1$$

We can now evaluate $\int_0^3 t^2 \sqrt{1+t} dt$ by carrying out the substitution suggested by (4) and (5). Thus

$$(6) \quad \begin{aligned} \int_0^3 t^2 \underbrace{\sqrt{1+t}}_{ds} dt &= \int_{s(0)}^{s(3)} \left(\left(\frac{3s}{2} + 1\right)^{2/3} - 1 \right)^2 ds \\ &= \int_0^{14/3} \left(\left(\frac{3s}{2} + 1\right)^{4/3} - 2 \left(\frac{3s}{2} + 1\right)^{2/3} + 1 \right) ds \\ &= \vdots \\ &= \left(\frac{2}{7}(8)^{7/3} - \frac{4}{5}(8)^{5/3} + \frac{14}{3} \right) - \left(\frac{2}{7}(1) - \frac{4}{5}(1) + 0 \right) \\ &= \frac{1696}{105} \end{aligned}$$

as we saw above.

Now let $g(s) = \left(\frac{3s}{2} + 1\right)^{2/3} - 1$, i.e., the integrand on the right-hand side of (6) above. We sketch the graph of $w = g(s)$ below.



The data points correspond to the third and last columns from the table on page 2. Notice that the integration carried out above yields the area under the curve $w = g(s)$.

Finally, the sketches below are a feeble attempt to show what the sheet would look like if the sheet was “unrolled” onto the xz -plane. Compare with the graph of $w = g(s)$ redrawn below.

