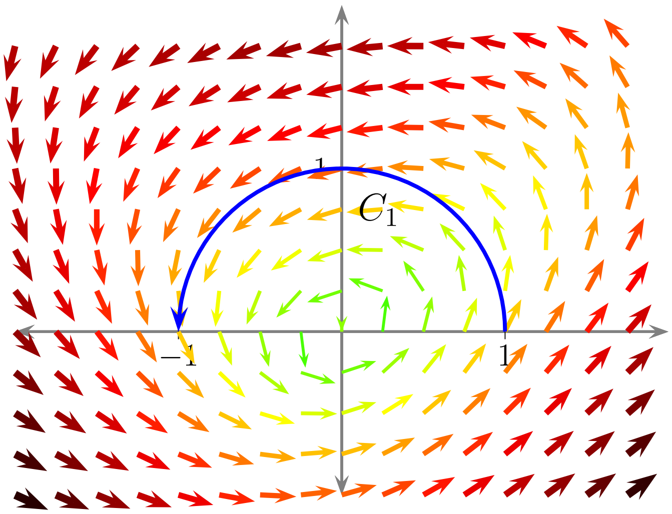

Figure 1: $\mathbf{F} = (x^2 - 3y)\ \mathbf{i} + (2x - \sin y)\ \mathbf{j}$

Example 1. Let $F(x,y) = \left\langle x^2 - 3y, 2x - \sin y \right\rangle$ and let C_1 be given by the vector equation

$$\mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}, \quad 0 \le t \le \pi. \tag{1}$$

Evaluate the flow integral

$$\int_{C_1} \mathbf{F}(x,y) \cdot d\mathbf{r} \tag{2}$$

This does not appear to be a pleasant integral to work with. Nevertheless, let's try. So

$$\begin{aligned} x &= \cos t, \quad & dx &= -\sin t\,dt \\ y &= \sin t, \quad & dy &= \cos t\,dt \end{aligned}$$

It follows that

$$\begin{aligned} M &= x^2 - 3y = \cos^2 t - 3\sin t \\ N &= 2x - \sin y = 2\cos t - \sin(\sin t) \end{aligned}$$

and

$$\begin{aligned} \mathbf{F} \cdot d\mathbf{r} &= M\,dx + N\,dy \\ &= \left[\left(3\sin t - \cos^2 t \right) \sin t + (2\cos t - \sin(\sin t)) \cos t \right] dt \end{aligned}$$

It follows that

$$\int_{C_1} \mathbf{F}(x,y)\,d\mathbf{r} = \int_0^{\pi} \Big[\big(3\sin t - \cos^2 t\big)\sin t + (2\cos t - \sin(\sin t))\cos t\Big]\,dt$$

$$= \underbrace{\int_0^{\pi} \big(3\sin t - \cos^2 t\big)\sin t\,dt}_{I_1} + \underbrace{\int_0^{\pi} (2\cos t - \sin(\sin t))\cos t\,dt}_{I_2} \tag{3}$$

Continuing we have

$$\begin{aligned} I_1 &= 3\int_0^{\pi} \sin^2 t\,dt - \int_0^{\pi} \cos^2 t\,\sin t\,dt \\ &= \frac{3\pi}{2} + \int_1^{-1} u^2\,du \\ &= \frac{3\pi}{2} - \frac{2}{3} \end{aligned}$$

and

$$\begin{aligned} I_2 &= 2\int_0^{\pi} \cos^2 t\,dt - \int_0^{\pi} \sin(\sin t)\cos t\,dt \\ &= \pi - \int_0^{0} \sin u\,du \\ &= \pi - 0 \end{aligned}$$

Putting this all together we see that

$$\int_{C_1} \mathbf{F}(x,y)\,d\mathbf{r} = \frac{3\pi}{2} - \frac{2}{3} + \pi = \frac{5\pi}{2} - \frac{2}{3}$$

Even though the last few calculations were not difficult, it would be nice if there was an easier way. For example, if the given vector field was conservative, then we could just find a potential function for **F** and invoke the Fundamental Theorem of Line Integrals. However, a quick inspection will reveal that the given field is not conservative.

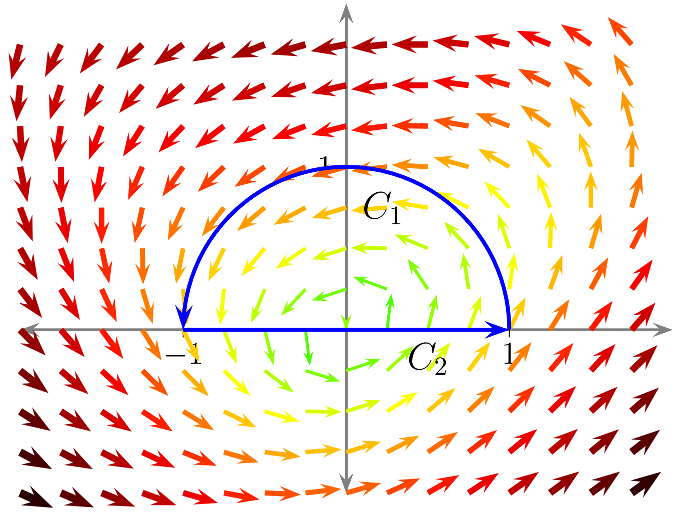

Figure 2: Green's Theorem

Instead we create a situation which allows us to use Green's Theorem. Let C_2 be the line segment from $(-1, 0)$ to $(1, 0)$. It might be easier to find the counterclockwise circulation around the closed curve $C = C_1 \cup C_2$ (see Figure 2). So let R be the region enclosed by C. Then by the Tangential Form of Green's Theorem,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA \\ &= \iint_R (2 - (-3))\, dA \\ &= 5 \times \left(\frac{1}{2} \text{area of unit circle} \right) \\ &= \frac{5\pi}{2}\end{aligned}$$

But how does that help us?

Notice that

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_C \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \\ &= \frac{5\pi}{2} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}\end{aligned}$$

So it would appear that we have traded one tedious calculation (integrating along

C_1) with another equally tedious calculation (integrating along C_2). However, a careful inspection will show that the last integral is much easier to compute.

Let $\mathbf{r}_2(t) = t\,\mathbf{i},\ -1 \le t \le 1$. Then $d\mathbf{r} = \mathbf{i}\,dt$ and

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{1} \left(t^2 - 3(0)\right)\, dt = 2/3$$

It follows that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \frac{5\pi}{2} - 2/3$$

as we saw above.

Let's try another one.

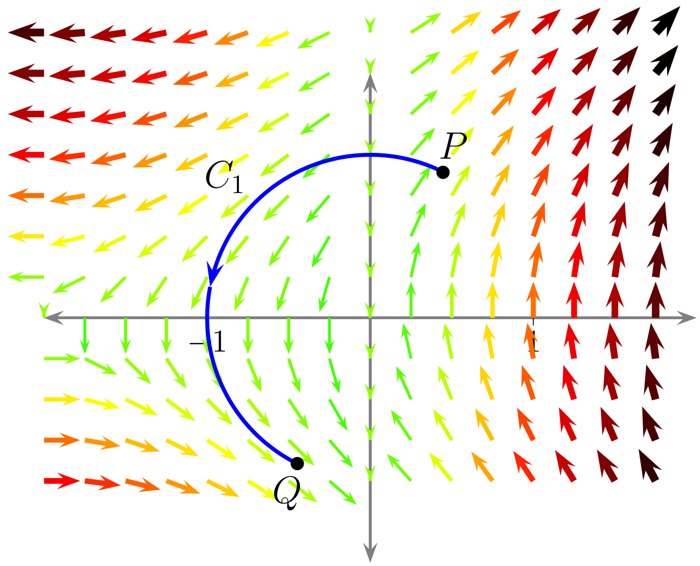

Figure 3: $\mathbf{F} = 2xy\,\mathbf{i} + (x^2 + bx)\,\mathbf{j}$

Example 2. Let $a, b > 0$ and $\mathbf{F} = 2xy\,\mathbf{i} + (x^2 + bx)\,\mathbf{j}$. Also let C_1 be given by the vector equation

$$\mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}, \quad \tan^{-1} a \leq t \leq \tan^{-1} a + \pi \tag{4}$$

Evaluate the flow integral

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \tag{5}$$

In other words, find the work done by a particle moving along the unit circle from P to Q (see Figure 3). Here

$$P = P\left((1 + a^2)^{-1/2}, a(1 + a^2)^{-1/2}\right)$$
$$Q = P\left(-(1 + a^2)^{-1/2}, -a(1 + a^2)^{-1/2}\right)$$

Using our usual notation we have $M = 2xy$ and $N = x^2 + bx$ and that

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x + b - 2x = b > 0$$

So if we could use Green's Theorem, the calculation of the circulation integral would be trivial.

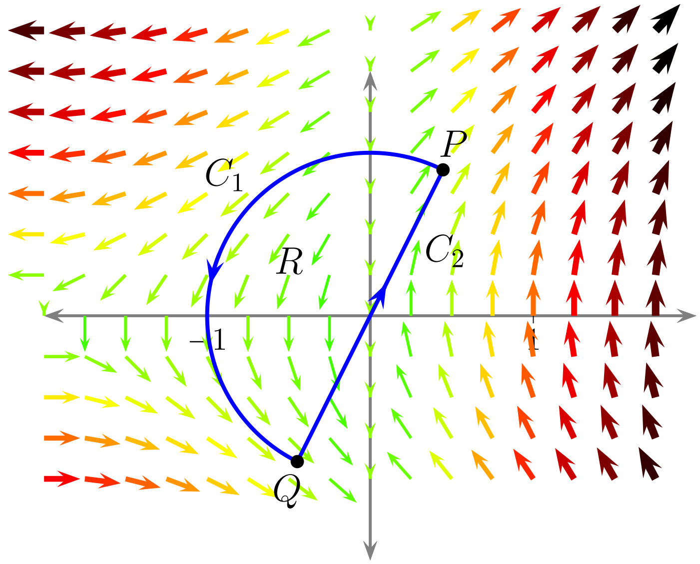

Figure 4: Another Application of Green's Theorem

So let C_2 be the straight line segment from Q to P and let R represent the region enclosed by the closed loop $C = C_1 \cup C_2$ (see Figure 4). Then R is a half-disk (of the unit circle) and by Green's Theorem

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int\int_C \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA \\ &= \int\int_C b \, dA \\ &= b \times \text{the area of } R \\ &= \frac{\pi b}{2}\end{aligned}$$

It follows that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \frac{\pi b}{2} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

We leave as an exercise to show that

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \frac{2a}{(1+a^2)^{3/2}}$$

It follows that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \frac{\pi b}{2} - \frac{2a}{(1+a^2)^{3/2}} \tag{6}$$

There is much we can learn from this example. Notice that if $b=0$ then $\mathbf{F}$ is a conservative vector field with potential function $f(x,y)=x^2y$. It follows that

$$\begin{aligned}\int_{C_1}\mathbf{F}\cdot d\mathbf{r} &= \int_P^Q df\\ &= x^2y\,\Big|_P^Q\\ &= \frac{-2a}{(1+a^2)^{3/2}}\end{aligned}$$

in agreement with (6).

Finally, we compute the integral below directly.

$$\begin{aligned}\int_{C_1}\mathbf{F}\cdot d\mathbf{r} &= \int_{C_1} M\,dx + N\,dy\\ &= \underbrace{-2\int_{\tan^{-1}a}^{\pi+\tan^{-1}a}\cos t\sin^2 t\,dt}_{I_1} + \underbrace{\int_{\tan^{-1}a}^{\pi+\tan^{-1}a}(\cos^2 t + b\cos t)\cos t\,dt}_{I_2}\end{aligned}$$

Now

$$\begin{aligned} I_1 &= \frac{-2\sin^3 t}{3}\ \Big|_{\tan^{-1} a}^{\pi+\tan^{-1} a} \\ &= \frac{4a^3}{3(1+a^2)^{3/2}} \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_{\tan^{-1} a}^{\pi+\tan^{-1} a} \cos^3 t\,dt + b\int_{\tan^{-1} a}^{\pi+\tan^{-1} a} \cos^2 t\,dt \\ &= \int_{\tan^{-1} a}^{\pi+\tan^{-1} a} (1-\sin^2 t)\cos t\,dt + \frac{\pi b}{2} \\ &= \int_{\frac{a}{\sqrt{1+a^2}}}^{\frac{-a}{\sqrt{1+a^2}}} (1-u^2)\,du + \frac{\pi b}{2} \\ &= \left(u - \frac{u^3}{3}\right)\ \Big|_{\frac{a}{\sqrt{1+a^2}}}^{\frac{-a}{\sqrt{1+a^2}}} + \frac{\pi b}{2} \\ &= \frac{-2a(3+2a^2)}{3(1+a^2)^{3/2}} + \frac{\pi b}{2} \end{aligned}$$

It follows that

$$\begin{aligned} \int_{C_1} \mathbf{F}\cdot d\mathbf{r} &= I_1 + I_2 \\ &= \frac{4a^3}{3(1+a^2)^{3/2}} + \frac{-2a(3+2a^2)}{3(1+a^2)^{3/2}} + \frac{\pi b}{2} \\ &= \frac{\pi b}{2} - \frac{2a}{(1+a^2)^{3/2}} \end{aligned}$$

as we saw above.