11.2 Infinite Series

Series and Partial Sums

Example 1. Infinite Sums

Find the following “sums”.

a) \[ \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \]

b) \[ \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots \]

c) \[ \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \cdots \]

d) \[ \sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + \cdots \]
What does it mean to add up an infinite number of things?

**Definition.** **Infinite Series**

An **infinite series** is the sum of an infinite sequence of numbers. Formally, it is

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n$$

For the remainder of this chapter whenever we use the term **series** it should be understood that we are referring to an **infinite series**.

**Remark.** Warning: Proceed with care when you see the word *formally* in mathematics. Loosely speaking it means “we are writing an expression that may or may not make any sense!”. For example, regardless of any subsequent definitions, the following series does not exist as a real or extended real number as we shall see later.

$$(1) \quad 1 - 1 + 1 - 1 + \cdots + (-1)^{n+1} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} = \sum_{n=0}^{\infty} (-1)^n$$
Definition. Infinite Series, nth Term, Partial Sum, etc.

Given the infinite series

\[ \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots \]  

we define the following. The number \( a_n \) is called the nth term of the series. It is also called the summand. The nth partial sum of the series is denoted by \( s_n \) and is defined by

\[
\begin{align*}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
& \vdots \\
s_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k \\
& \vdots
\end{align*}
\]

Notice that the partial sums generate a new sequence, the so-called sequence of partial sums, \( \{s_n\} \). Now if this new sequence converges to a limit, say \( L \in \mathbb{R} \), we say that the series (2) converges and that its sum is \( L \). Specifically,

\[ s_n \to L \text{ as } n \to \infty \implies \sum_{n=1}^{\infty} a_n = L \]  

In other words,

\[ \sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \sum_{k=1}^{n} a_k = \lim_{n \to \infty} s_n \]

whenever the limit exists. Otherwise, the series diverges.
Note: We sometimes drop the indices when it is convenient. In such cases, \( \sum a_n \) is understood to mean \( \sum_{n=1}^{\infty} a_n \) whether or not the series converges.

Example 2. Does the series below converge or diverge.

\[
\sum_{n=1}^{\infty} \frac{1}{n(n + 1)}
\]

We claim that the series converges. Using partial fractions, we first rewrite the summand as \( 1/n - 1/(n + 1) \). Thus

\[
s_n = \sum_{j=1}^{n} \frac{1}{j(j+1)} = \sum_{j=1}^{n} \left( \frac{1}{j} - \frac{1}{j+1} \right)
\]

\[
= \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \cdots + \left( \frac{1}{n} - \frac{1}{n+1} \right)
\]

\[
= 1 + \left( \frac{1}{2} - \frac{1}{2} \right) + \left( \frac{1}{3} - \frac{1}{3} \right) + \cdots + \left( \frac{1}{n} - \frac{1}{n} \right) - \frac{1}{n+1}
\]

\[
= 1 - \frac{1}{n+1}
\]

It follows that the series converges. In fact,

\[
\sum_{n=1}^{\infty} \frac{1}{n(n + 1)} = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left( 1 - 1/(n + 1) \right) = 1
\]

Remark. In this example we took advantage of something called a
telescoping sum. In general, a **telescoping sum** is a series of the form

\[
\sum_{j=1}^{n} (a_j - a_{j+1}) = (a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \cdots + (a_n - a_{n+1})
\]

\[
= a_1 + (a_2 - a_2) + (a_3 - a_3) + \cdots + (a_n - a_n) - a_{n+1}
\]

\[
= a_1 - a_{n+1}
\]

Now suppose that the sequence \( \{a_n\} \) is convergent. That is, suppose that \( a_n \to a \) as \( n \to \infty \). Then

\[
\sum_{n=1}^{\infty} (a_n - a_{n+1}) = \lim_{n \to \infty} (a_1 - a_{n+1}) = a_1 - a
\]
Geometric Series

A **geometric series** is a series of the form

\[
a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n
\]

where \(a\) and \(r\) are fixed constants with \(a \neq 0\). The constant \(r\) is usually called the **common ratio**.

We wish to obtain a closed formula for (5). Suppose that the series in (5) converges to a real number, call it \(s\). Then

\[
s = \sum_{n=0}^{\infty} ar^n = a + \sum_{n=0}^{\infty} ar^{n+1}
\]

\[
= a + r \sum_{n=0}^{\infty} ar^n = a + rs
\]

Thus

\[
\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}
\]

Now the right-hand side of (7) is defined for all \(r \neq 1\). On the other hand, it is easy to see that the left-hand side of (7) diverges for \(|r| > 1\) (Why?). It appears that a bit more care is needed.

Instead, we consider the \(n\)th partial sum of \(\sum_{k=0}^{\infty} r^k\).

\[
s_n = 1 + r + r^2 + \cdots + r^n
\]

\[
\Rightarrow \quad rs_n = r + r^2 + r^3 + \cdots + r^{n+1}
\]
Now subtract the second row from the first to obtain
\[ s_n - rs_n = 1 - r^{n+1} \quad \text{or} \]
\[ s_n = \frac{1 - r^{n+1}}{1 - r} \]

Now suppose that \(|r| < 1\). Then, by Theorem 11.1.5 (Common Limits Theorem), \(r^{n+1} \to 0\) as \(n \to \infty\) and
\[ 1 + r + r^2 + \cdots + r^n + \cdots \text{converges to} \quad \frac{1}{1 - r} \tag{8} \]
In general, we have
\[ \sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}, \quad |r| < 1. \tag{9} \]
If \(|r| \geq 1\) then the series diverges.

**Example 3.** Find the following (infinite) sum...if it exists.
\[ \sum_{n=0}^{\infty} 5 \left( \frac{1}{3} \right)^n \]
Notice that the series is geometric with common ratio \(1/3\). From (9) we conclude that
\[ \sum_{n=0}^{\infty} 5 \left( \frac{1}{3} \right)^n = \frac{5}{1 - 1/3} \]

**Example 4.** Express \(2.\overline{325}\) as a ratio of two integers.
The Divergence Test

Notice that whenever \( \sum a_n \) converges the terms \( a_n \) must approach 0. To see this, let \( \{ s_n \} \) be the partial sums of the infinite series \( \sum a_n \). That is, let

\[
s_n = \sum_{k=0}^{n} a_n
\]

and suppose that

\[
\sum_{n=0}^{\infty} a_n = L, \quad L \in \mathbb{R}
\]

Then

\[
\lim_{n \to \infty} s_n = L
\]

Notice that \( a_n = s_n - s_{n-1} \). It follows that

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} (s_n - s_{n-1})
\]

\[
= \lim_{n \to \infty} s_n - \lim_{n \to \infty} s_{n-1}
\]

\[
= L - L
\]

\[
= 0
\]

We have

**Theorem 1.** If \( \sum a_n \) converges then \( a_n \to 0 \) as \( n \to \infty \).
Remark. The converse is not true. That is, there are infinite series whose terms go to zero but the series fails to converge. Consider the example below.

Example 5. The Harmonic Series Diverges

That is

\[
\sum_{n=1}^{\infty} \frac{1}{n} = \infty
\]

To see this, notice that

\[
\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{15} + \frac{1}{16} + \cdots
\]

\[
= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \frac{1}{9} + \cdots + \frac{1}{16} + \cdots
\]

\[
> \frac{3}{2} + \frac{1}{4} + \frac{1}{4} + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \frac{1}{16} + \cdots + \frac{1}{16} + \cdots
\]

\[
= \frac{3}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots
\]

In other words, the sequence of partial sums is increasing without bound and \((10)\) is established.
Here’s shorter proof. It is easy to show that if $x > 1$, one has

\[
\frac{1}{x - 1} + \frac{1}{x} + \frac{1}{x + 1} > \frac{3}{x}
\]

Exercise: Verify this.

No suppose that the harmonic series converged, say to some real number $s$. Then

\[
s = \sum_{n=1}^{\infty} \frac{1}{n}
\]

\[
= 1 + \left( \frac{1}{3 - 1} + \frac{1}{3} + \frac{1}{3 + 1} \right) + \left( \frac{1}{6 - 1} + \frac{1}{6} + \frac{1}{6 + 1} \right) + \cdots
\]

\[
> 1 + 3 \left( \frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \cdots \right) = 1 + \left( \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \right)
\]

\[
= 1 + \sum_{n=1}^{\infty} \frac{1}{n}
\]

\[
= 1 + s
\]

This is absurd. We conclude that the harmonic series must diverge.
In the next section we will give another proof that the harmonic series diverges.

**The nth-Term Test for Divergence (the Divergence Test)**

If \( \lim_{n \to \infty} a_n \neq 0 \) then the series \( \sum_{n=0}^{\infty} a_n \) diverges.

**Note:** This is the *contrapositive* of Theorem \(^1\)

For example, the series \( \sum_{n=1}^{\infty} \frac{n}{2n + 1} \) diverges since

\[
\lim_{n \to \infty} \frac{n}{2n + 1} = \frac{1}{2}
\]

What does the nth-Term Test for Divergence say about the series

\[
\sum_{n=1}^{\infty} \frac{|\sin n|}{n}
\]

Nothing! Since \( \frac{|\sin n|}{n} \to 0 \) as \( n \to \infty \), the test does not apply.

Do not underestimate the usefulness of the Divergence Test (and of Theorem \(^1\)).

**Example 6.** Find the sum or show that the series diverges.

\[
\sum_{n=1}^{\infty} \ln \frac{n}{2n + 1}
\]
The following theorem is a direct consequence of Theorem 2 from section 11.1.

**Theorem 2. Combining Series**

If \( \sum a_n = A \) and \( \sum b_n = B \) are convergent series, then

1. **Sum-Difference Rule**: \( \sum (a_n \pm b_n) = \sum a_n \pm \sum b_n = A \pm B \)

2. **Constant Multiple Rule**: \( \sum c a_n = c \sum a_n = cA \) for any real number \( c \).
Example 7. Find the sum.

\[ \sum_{n=0}^{\infty} \frac{1 - 2^{n-1}}{4^n} = \sum_{n=0}^{\infty} \frac{1}{4^n} - \sum_{n=0}^{\infty} \frac{2^{n-1}}{4^n} \]

\[ = \frac{1}{1 - 1/4} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{2^n}{4^n} \]

\[ = \frac{4}{3} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} \]

\[ = \frac{4}{3} - \frac{1}{2} \frac{1}{1 - 1/2} \]

\[ = \frac{1}{3} \]
Remark. If $\sum a_n = \infty$, i.e., if the series $\sum a_n$ diverges to infinity, then we can still use the constant multiple rule provided we are careful. In particular, we must avoid indeterminate forms such as $0 \times \infty$ or $\infty - \infty$.

For example, if $c \neq 0$ we can apply the constant multiple rule to conclude that $\sum c a_n$ diverges whenever $\sum a_n$ does.

For example, 
\[
\sum_{n=1}^{\infty} \frac{2}{n} = 2 \sum_{n=1}^{\infty} \frac{1}{n} = 2 \times \infty = \infty
\]

So the series diverges.

Cesaro Summability - Increasing the No. of Convergent Series?

We begin with a curious example. Suppose that the series in (1) did converge to a real number $s$. Then 
\[
s = \sum_{n=0}^{\infty} (-1)^n
\]
\[
= 1 - 1 + 1 - 1 + \cdots
\]
\[
= 1 - (1 - 1 + 1 - 1 + \cdots)
\]
\[
= 1 - s
\]

It follows that 
\[
\sum_{n=0}^{\infty} (-1)^n = 1/2
\]

Of course, this is ridiculous since the series diverges by the nth term test.
Nevertheless, observations such the one given above often have merit as we shall see later. We seek a method to increase the number of “convergent” series.
Given a series $\sum a_n$ and its associated sequence of partial sums $s_n = \sum_{j=0}^{n} a_j$. We define a new sequence, the so-called Cesaro sum by

$$\sigma_n = \sum_{j=0}^{n-1} \left(1 - \frac{j}{n}\right) a_j = \frac{s_0 + s_1 + \cdots + s_{n-1}}{n} = \frac{1}{n} \sum_{j=0}^{n-1} s_j$$

Note: Cesaro sums represents an “averaging” process. In 1890 the Italian mathematician Ernesto Cesaro used such sums while investigating products of infinite series. This technique was also used with much success by Humphrey Bogart to land the starring role in several notable films, including “Casablanca” and “The Big Sleep”.

**Definition. Cesaro Summability**

A series $\sum a_n$ is called Cesaro summable if its Cesaro sums converge. That is, if

$$\lim_{n \to \infty} \sigma_n = L \in \mathbb{R}$$

**Example 8.** Let’s compute the Cesaro sums of the divergent series from (1). The partial sums are $s_{2n} = 1$ and $s_{2n+1} = 0$. It follows that

$$\sigma_{2n+1} = \frac{1}{2n+1} (1 + 0 + 1 + 0 + \cdots + 1) = \frac{n + 1}{2n + 1}$$

$$\sigma_{2n} = \frac{1}{2n} (1 + 0 + 1 + \cdots + 0) = \frac{1}{2}$$

Hence

$$\lim_{n \to \infty} \sigma_{2n} = \lim_{n \to \infty} \sigma_{2n+1} = 1/2$$
It follows that the divergent series in (1) is Cesaro summable to $1/2$.

The next theorem shows that Cesaro summable series converge to the “right” limit whenever the (original) series converges.

**Theorem 3.** Suppose that $\sum a_n$ is a convergent series with sum, say $L$. Then $\sum a_n$ is Cesaro summable to $L$. Specifically, let $s_n = \sum_{j=0}^{n} a_j$. Then

(14) \[ \lim_{n \to \infty} s_n = L \quad \implies \quad \lim_{n \to \infty} \sigma_n = L \]
Proof. Let $\varepsilon > 0$. So there is a positive integer $N$ such that $n \geq N$ implies $|s_n - L| < \varepsilon$. We have

$$|\sigma_{N+n} - L| = \left| \frac{1}{N+n} \sum_{j=0}^{N+n-1} s_j - \frac{N+n}{N+n} L \right|$$

$$= \frac{1}{N+n} \left| \sum_{j=0}^{N+n-1} s_j - \sum_{j=0}^{N+n-1} L \right|$$

$$\leq \frac{1}{N+n} \sum_{j=0}^{N+n-1} |s_j - L|$$

$$= \frac{1}{N+n} \left( \sum_{j=0}^{N-1} |s_j - L| + \sum_{j=N}^{N+n-1} |s_j - L| \right)$$

less than $\varepsilon$

$$\leq \frac{1}{N+n} \left( \sum_{j=0}^{N-1} |s_j - L| + n \varepsilon \right)$$

$$= \frac{1}{N+n} \sum_{j=0}^{N-1} |s_j - L| + \frac{n \varepsilon}{N+n}$$

$$< \frac{1}{N+n} \sum_{j=0}^{N-1} |s_j - L| + \varepsilon$$

Independent of $n$

Now let $n \to \infty$ and the result follows. \qed
To reiterate, the theorem shows that convergent series are necessarily Cesaro summable and the Cesaro sum is equal to the original limit. However, the converse is not true as we saw in Example 8.

We finish with a curious follow-up to Example 8. Recall that under questionable reasoning one might conclude that the divergent series from (1) “converges” to 1/2. In fact, this was debated in Euler’s time (see Guido Grandi’s 1703 paper). It was Cesaro and his contemporaries that added rigor to such a conclusion by defining new types of convergence criteria. As we mentioned earlier, these were called summability methods.

As we saw above, we now can say that the divergent series $\sum_{n=0}^{\infty}(-1)^n$ is Cesaro summable to 1/2.

Now consider the product $(1 - 1 + 1 - 1 + \cdots)^2$. It is not unreasonable to argue that

$$\sum_{n=0}^{\infty}(-1)^n$$

(15) $(1 - 1 + 1 - 1 + \cdots)^2 \overset{?}{=} (1/2)^2 = 1/4$

and to justify such a conclusion using our new summability methods. That is, we should be able to show that $(1 - 1 + 1 - 1 + \cdots)^2$ is Cesaro summable to 1/4. Unfortunately,

$$(1 - 1 + 1 - 1 + \cdots)^2 = (1 - 1 + 1 - 1 + \cdots) \times (1 - 1 + 1 - 1 + \cdots)$$

$$= 1 - 2 + 3 - 4 + 5 + \cdots$$

$$= \sum_{n=0}^{\infty}(-1)^n n$$

is not Cesaro summable (to anything). It turns out that the series is Abel summable to 1/4. We will have more to say about this example and other types of summability in section 11.7.
Example 9. Show that the formula above for \((1 - 1 + 1 - 1 + \cdots)^2\) is valid. Also, show that its Cesaro sums \(\sigma_n\) diverge by showing \(\sigma_n \to 1/2\) or \(-1/2\) depending on the parity of \(n\). We leave this as an exercise.