7.2 Trigonometric Integrals

Products of Powers of Sines and Cosines

We wish to evaluate integrals of the form:

\[ \int \sin^m x \cos^n x \, dx \]

where \( m \) and \( n \) are nonnegative integers.

Recall the double angle formulas for the sine and cosine functions.

\[
\begin{align*}
sin 2x &= 2 \sin x \cos x \\
\cos 2x &= \cos^2 x - \sin^2 x \\
&= 2 \cos^2 x - 1 \\
&= 1 - 2 \sin^2 x
\end{align*}
\]

The cosine formulas can be used to derive the very important “trig reduction” formulas.

(1) \[ \cos^2 x = \frac{1}{2} (1 + \cos 2x) \]

(2) \[ \sin^2 x = \frac{1}{2} (1 - \cos 2x) \]
Equations (1) and (2) are very useful when integrating *even* powers of sine and cosine.

**Example 1.** Evaluate the following integrals.

a. \[ \int \cos^2 x \, dx \]

   By (1) we have

   \[
   \int \cos^2 x \, dx = \frac{1}{2} \int (1 + \cos 2x) \, dx \\
   = \frac{1}{2} \left( x + \frac{\sin 2x}{2} \right) + C
   \]

b. \[ \int \sin^4 x \, dx \]

   Using reduction (twice) we have

   \[
   \sin^4 x = \left( \sin^2 x \right)^2 \\
   = \frac{1}{4} (1 - \cos 2x)^2 \\
   = \frac{1}{4} \left( 1 - 2 \cos 2x + \cos^2 2x \right) \\
   = \frac{1}{4} \left( 1 - 2 \cos 2x + \frac{1}{2} (1 + \cos 4x) \right) \\
   = \frac{1}{8} \left( 3 - 4 \cos 2x + \cos 4x \right)
   \]
It follows that
\[
\int \sin^4 x \, dx = \frac{1}{8} \int (3 - 4 \cos 2x + \cos 4x) \, dx \\
= \frac{1}{8} \left( 3x - 2 \sin 2x + \frac{\sin 4x}{4} \right) + C
\]

Example 2. Even Products

Evaluate
\[
\int \sin^4 x \cos^6 x \, dx
\]

This is not too difficult since
\[
\sin^4 x \cos^6 x = (\sin^2 x)^2 \cos^6 x \\
= (1 - \cos^2 x)^2 \cos^6 x \\
= (1 - 2 \cos^2 x + \cos^4 x) \cos^6 x \\
= \cos^6 x - 2 \cos^8 x + \cos^{10} x
\]

Thus
\[
\int \sin^4 x \cos^6 x \, dx \\
= \int \cos^6 x \, dx - 2 \int \cos^8 x \, dx + \int \cos^{10} x \, dx
\]

and we can proceed as before (to handle the odd powers that appear, see Example 3 below).

For example, to integrate the last term above, we expand
\[
32 \cos^{10} x = (1 + \cos 2x)^5 \\
= 1 + 5 \cos 2x + 10 \cos^2 2x + 10 \cos^3 2x + 5 \cos^4 2x + \cos^5 2x
\]
Now the odd-powered terms are easy to handle (see below). For the even-powered terms, we must repeat the trig reduction formulas (1) and (2) as many times as necessary to obtain a workable integrand.

Thus

\[ 32 \cos^{10} x = (1 + \cos 2x)^5 \]

\[ = 1 + 5 \cos 2x + 10 \cos^2 2x + 10 \cos^3 2x + 5 \cos^4 2x + \cos^5 2x \]

\[ = 1 + 5 \cos 2x + 5(1 + \cos 4x) \]

\[ + 10 \cos^3 2x + \frac{5}{4}(1 + \cos 4x)^2 + \cos^5 2x \]

\[ = 1 + 5 \cos 2x + 5(1 + \cos 4x) \]

\[ + 10 \cos^3 2x + \frac{5}{4}(1 + 2 \cos 4x + \cos^2 4x) + \cos^5 2x \]

\[ = \frac{29}{4} + 5 \cos 2x + 5 \cos 4x + 10 \cos^3 2x + \frac{5}{2} \cos 8x + \frac{5}{4}(\cos^2 8x) + \cos^5 2x \]
Odd powers are somehow easier.

**Example 3.** Evaluate

\[
\int \cos^3 4x \, dx
\]

\[
\int \cos^3 4x \, dx = \int \cos^2 4x \cos 4x \, dx
\]

\[
= \int (1 - \sin^2 4x) \cos 4x \, dx
\]

Now let \( u = \sin 4x \). Then \( du = 4 \cos 4x \, dx \) and

\[
= \frac{1}{4} \int (1 - u^2) \, du
\]

\[
= \frac{1}{4} \left( u - \frac{u^3}{3} \right) + C
\]

\[
= \frac{1}{4} \left( \sin 4x - \frac{\sin^3 4x}{3} \right) + C
\]
7.2

Example 4. Evaluate

\[ \int \sin^4 x \cos^5 x \, dx \]

Guided by example 2, we might try

\[ = \int \sin^4 x (\cos^2 x)^2 \cos x \, dx \]

\[ = \int \sin^4 x (1 - \sin^2 x)^2 \cos x \, dx \]

Now we let \( u = \sin x \). Then

\[ = \int u^4 (1 - u^2)^2 \, du \]

\[ = \int (u^4 - 2u^6 + u^8) \, du \]

\[ = \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + C \]

\[ = \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + C \]
Integrals of Powers of Tangent and Secant

Example 5. Powers of Secant

Evaluate $\int \sec^3 x \, dx$.

We try integration by parts.

\[ u = \sec x \quad \quad \quad dv = \sec^2 x \, dx \]
\[ du = \sec x \tan x \, dx \quad \quad \quad v = \tan x \]

Thus

\[ \int \sec^3 x \, dx = \sec x \tan x - \int \tan^2 x \sec x \, dx \]
\[ = \sec x \tan x + \int \sec x \, dx - \int \sec^3 x \, dx \]

and this looks familiar. As we saw earlier, we can now “solve for the integral”.

\[ 2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx \]
\[ = \sec x \tan x + \ln |\sec x + \tan x| + C \]

Thus

\[ \int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C \]
Products to Sum Formulas

Recall the addition and subtraction formulas for sine and cosine.

(3) \[ \sin(u \pm v) = \sin u \cos v \pm \sin v \cos u \]
(4) \[ \cos(u \pm v) = \cos u \cos v \mp \sin u \sin v \]

For example, using (4) one can find the exact value of \( \cos \pi/12 \).

\[
\begin{align*}
\cos \frac{\pi}{12} &= \cos \left( \frac{\pi}{3} - \frac{\pi}{4} \right) \\
&= \cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4} \\
&= \frac{1}{2} \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \frac{\sqrt{2}}{2} \\
&= \frac{1}{2} \left( \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \right)
\end{align*}
\]

Now let \( m \) and \( n \) be real numbers. Notice that

\[
\sin(m - n)x = \sin mx \cos nx - \sin nx \cos mx
\]
and

\[
\sin(m + n)x = \sin mx \cos nx + \sin nx \cos mx
\]

Which implies that

\[
\sin(m - n)x + \sin(m + n)x = 2 \sin mx \cos nx
\]

Now we can evaluate integrals of the form \( \int \sin mx \cos nx \, dx \). For example,

\[
\begin{align*}
\int \sin 5x \cos 3x \, dx &= \frac{1}{2} \int \sin(5 + 3)x + \sin(5 - 3)x \, dx \\
&= -\frac{1}{2} \left( \frac{\cos 8x}{8} + \frac{\cos 2x}{2} \right) + C
\end{align*}
\]
As we might expect, there are 3 product-to-sum formulas.

\begin{align*}
(5) \quad 2 \sin mx \sin nx &= \cos(m - n)x - \cos(m + n)x \\
(6) \quad 2 \sin mx \cos nx &= \sin(m - n)x + \sin(m + n)x \\
(7) \quad 2 \cos mx \cos nx &= \cos(m - n)x + \cos(m + n)x
\end{align*}

We derived (6) above. The other two identities are proven in a similar manner. These identities turn out to be important in \textbf{Fourier Analysis}. 
Example 6. The Orthogonality of Sine and Cosine on $[-\pi, \pi]$

Suppose $m$ and $n$ are positive integers. Then

\[
\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0
\]

\[
\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0, \quad m \neq n
\]

\[
\int_{-\pi}^{\pi} \sin mx \sin mx \, dx = \int_{-\pi}^{\pi} \cos mx \cos mx \, dx = \pi
\]

For example, $\sin mx \cos nx$ is an odd function hence

\[
\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0.
\]

Recall that $\sin k\pi = 0$ whenever $k$ is an integer. Then by (6) we have

\[
\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (\cos(mx - nx) - \cos(m + n)x) \, dx
\]

\[
= \frac{1}{2} \left( \frac{\sin(mx - nx)}{m - n} + \frac{\sin(m + n)x}{m + n} \right) \bigg|_{-\pi}^{\pi}
\]

\[
= 0
\]

since $m \pm n$ is an integer. The remaining identities are proven in a similar fashion.
In 1822 Jean Baptiste Joseph Fourier published *The Analytical Theory of Heat*. In chapter III he made the bold claim that an arbitrary function \( f(x) \) defined on the interval \([-\pi, \pi]\) could be expressed as an infinite trigonometric series of the form

\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx
\]

(8)

Although Daniel Bernoulli had earlier (c. 1740) suggested such a representation, he was unable to show how the coefficients could be computed. On the other hand, Fourier claimed that

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx
\]

and for \( n > 0 \),

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx
\]

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx
\]

It would take many years before Fourier's claims could be rigorously proven (albeit, for a much smaller class of functions). Today, Fourier series are an integral part of many undergraduate curriculums.

Exactly what do we mean by the infinite summations in (8)? Briefly, when an *infinite series*, such as third term in (8), exists as a real number, it is said to *converge*. In chapter 11 we will define and study the convergence of these infinite series.

**Example 7.** Compute the Fourier Coefficients for \( f(x) = x \)
Notice that $a_0$ is 0 since $f(x) = x$ is an odd function. Also, $x \cos nx$ is odd, hence $a_n = 0$ for all $n > 0$. On the other hand, if we let $u = x$ and $dv = \sin nx \, dx$, then partial integration yields

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx$$

$$= \frac{-1}{\pi} \frac{x \cos nx}{n} \bigg|_{-\pi}^{\pi} + \frac{1}{n\pi} \int_{-\pi}^{\pi} \cos nx \, dx$$

$$= \frac{-1}{n\pi} (\pi \cos n\pi + \pi \cos (-n\pi)) + 0$$

$$= \frac{2(-1)^{n+1}}{n}$$

In other words,

(9) \quad x = ? \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx

The right-hand side of (9) is called the Fourier (Sine) series of $f(x) = x$. The question mark is a reminder that we are not quite sure what is meant by the right-hand side of the equality.

Now let $N$ be a positive integer. Then the finite sum

(10) \quad S_N(x) = \sum_{n=1}^{N} \frac{2(-1)^{n+1}}{n} \sin nx

is called the $N$th partial sum (of the series (9)). For any $N > 0$ it is clear these finite sums exist. Below we plot a few of these partial sums for small values of $N$. 
It seems believable that if $x$ is somehow represented by the infinite series given in (9), then the 100th partial sum should be a better approximation than the 4th. The annoying artifact visible near $\pm \pi$ for $S_{100}(x)$ shows what can go wrong. It is known as Gibbs Phenomenon and it is unavoidable whenever one attempts to represent a non-periodic function, such as $f(x) = x$, using Fourier series. Over the years many researchers have expended much time and effort in an attempt to find alternative series representations of such functions in the hope of reducing these effects.