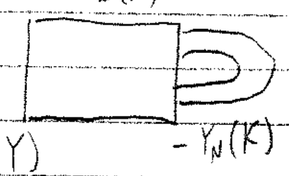
Matt Hedden HFH 2/22/11

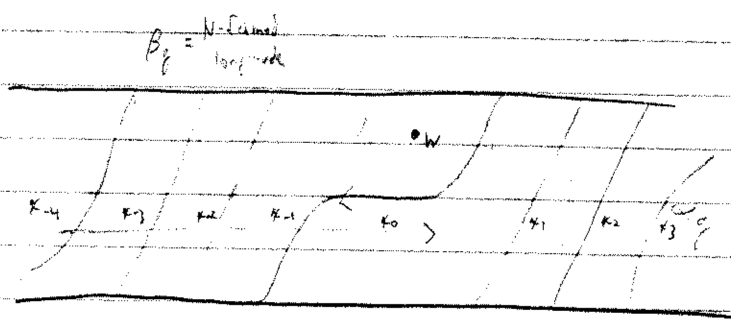
Thm Let $K \subseteq Y$ be a knot. Then $\forall N \gg 0$,

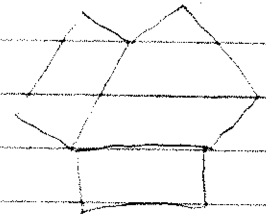
$$\left.\begin{array}{l} HF^+(\\ HF^-(\\ \widehat{HF}(\end{array} Y_N(K), s_m\right) \cong H_*\left(C\left(\max\{i, j-m\} \begin{array}{l}\geq 0 \\ < 0 \\ = 0\end{array}\right)\right)$$



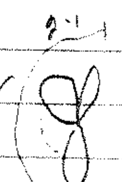
$$\partial W_N'(K) = -Y_N(K) \sqcup Y$$

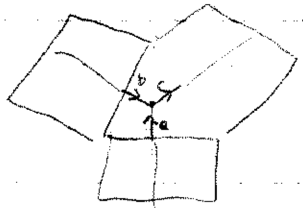
Neg. def. cobordism from $Y_N(K)$ to Y.

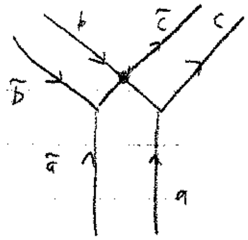


$W_{\alpha\beta\gamma} :=$  $= (\Delta \times \Sigma) \sqcup (U_\alpha \times I) \sqcup (U_\beta \times I) \sqcup (U_\gamma \times I) / \sim$

$$\partial W_{\alpha\beta\gamma} = -Y_N^3(K) \sqcup \#^{g-1} S^1 \times S^3 \sqcup Y$$

Claimed $W_{\alpha\beta\gamma} = -W_N'(K) \setminus \text{nbhd}\Big($  $\Big)$





triply periodic domain

sign($\tilde{s} \cap s$)

sign($\tilde{c} \cap b$) $\cdot$ sign($\partial_\gamma P \cap \partial_\beta P$)

$\ominus$ $\oplus$

Define a map

$$\Phi_m^- : CF^-(Y_N(K), s_m) \longrightarrow C(\max\{i, j-m\} < 0) \quad \text{by}$$

$$CF^- = \bigoplus_{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \mathbb{Z}[U]\vec{x} \overset{\text{identify}}{\leadsto} [x, i], \; i<0$$

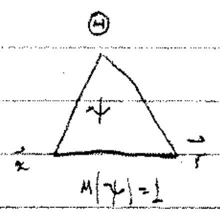
$$\Phi_m^-([x,i]) = \sum_{\vec{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma} \sum_{\substack{\psi \in \pi_2(\vec{x}, \Theta_{top}, \vec{y}) \\ \mu(\psi)=0}} \#\mathcal{M}(\psi) \cdot [\vec{y}, i - n_w(\psi), (m-1) - n_z(\psi)]$$

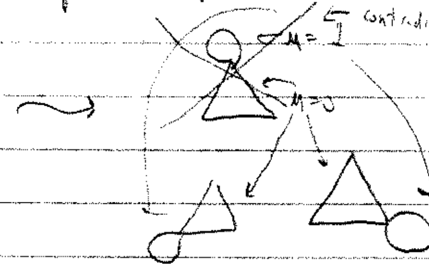
$$\star \; n_z(\psi) - n_w(\psi) = m - A(\vec{y})$$

Claims:

(1) Φ^- is a chain map

via Gromov compactness applied to each $\psi \in \pi_2(\vec{x}, \Theta, \vec{y})$ satisfying our conditions & $\mu(\psi) = 1$.





(2) Well-defined, i.e. is range of Φ^- contained in $C(\max\{i, j-m\} < 0)$?

Since ψ has J-hol reps., $n_z(\psi), n_w(\psi) \geq 0$

(3) Why is the domain of $CF^-(Y_N(K))$ restricted to s_m?
Claim is that $\textcircled{\$}$ forces ψ to connect $\vec{x}$ s.t. $s_w(\vec{x}) = s_m$.

To $\psi \in \pi_2(\vec{x}, \Theta, \vec{y})$, we associate $s_w(\psi) \in Spin^c(W_{\alpha\beta\gamma})$

Recall, $s_m \in Spin^c(Y_N(K))$ is defined by

1. s_m extends to t_m over $-W_N'(K)$ s.t. $t_m|_Y = s$
2. $\langle c_1(t_m), [S]\rangle + N = 2m$

Prop.

$$\langle c_1(s_w(\psi)), [S]\rangle + N = \langle c_1(\underline{s}(\vec{y})), [\hat{F}]\rangle + 2(n_z(\psi) - n_w(\psi))$$

Chern class of $Spin^c$-structure associated to ψ by basepoint map

Chern class of relative $Spin^c$-structure evaluated on capped off Seifert surface.

Assuming this Prop., we prove the above Claim.

Recall, $2A(\vec{y}) = \langle c_1(\underline{s}(\vec{y})), [\hat{F}]\rangle$

$$2\,\textcircled{\$} \Leftrightarrow 2(n_z(\psi) - n_w(\psi)) = 2m - 2A(\vec{y})$$
$$\Leftrightarrow 2m = \langle c_1, [F]\rangle + 2(n_z - n_w)$$

So RHS of Claim $= 2m$

$\Rightarrow \langle c_1(s_w(\psi)), [S]\rangle + N = 2m$.

Remark: Φ^- extends to a map $\Phi^\infty : CF^\infty(Y_N(K), s_m) \to CFK^\infty$

& Φ^- is restriction of $\Phi^\infty|_{CF^-}$.

(Φ^∞ is defined the same, but domain is $[x, i]$, $i \in \mathbb{Z}$)

$$\hat{\Phi} = \Phi^\infty|_{\widehat{CF}(Y_N(K), s_m)}$$

Want to show Φ^- induces $\cong$ on homology

Idea. We'll filter $CF^\infty(Y_N(K))$ + CFK^∞ by the symplectic area functional (i.e. $F^{Area}(\vec{x}) - F^{Area}(\vec{y}) = Area(D(\phi))$ for $\phi \in \pi_2(\vec{x}, \vec{y})$).

wrt these filtrations, Φ^- is a filtered chain map.

Since J-holomorphic $\psi \in \pi_2(\vec{x}, \Theta, \vec{y})$ have $Area(D(\psi)) > 0$.

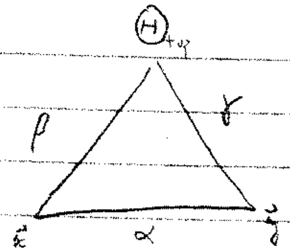
Just as in proof of handleslide / isotopy (ii) invariance, will see that
$$\hat{\Phi}^{\pm}_{\infty} = \underset{L}{\text{Isomorphism}} + \underset{H}{\text{higher order terms}}$$

Taking filtered basis gives
$$\Phi = \begin{bmatrix} 1 & & ? \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

Then by Linear Algebra, $\exists$ filtered change of basis so that
$$\Phi = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = Id. \quad \Rightarrow \Phi \text{ induces isomorphism on homology.}$$

Recall, there was an $N \to 1$ map
$$\vec{x} \in \text{winding region } \mathbb{T}_\alpha \cap \mathbb{T}_\beta \subseteq CF^\infty(Y_N(K)) \xrightarrow{\pi} \vec{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \subseteq CFK^\infty(Y, K)$$
$$\{x_i, \vec{r}\} \longmapsto \{x_0, \vec{r}\} \quad (= \vec{z})$$
$$i = -\tfrac{N}{2}, \dots \hat{0} \dots \tfrac{N}{2}$$

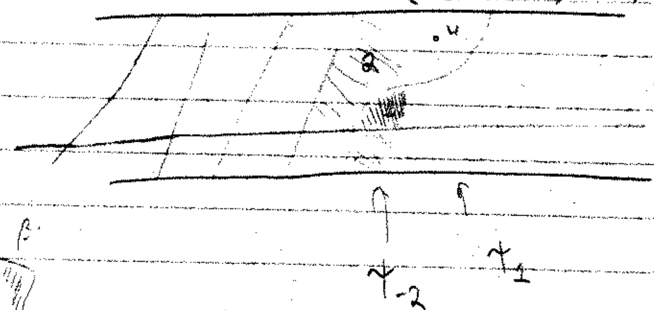
By focusing attention on $\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ s.t. $s_w(\vec{x}) = s_m$, we've restricted π to a single ξ-class, & it is a bijection.

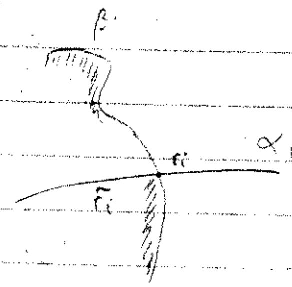


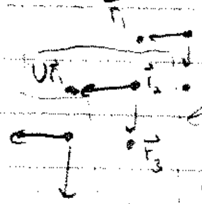
$\longrightarrow \left(\mathrm{Sym}^g, \mathbb{T}_\alpha, \mathbb{T}_\beta, \mathbb{T}_\gamma \right)$

For each $\{x_0, \vec{r}\}$, $\exists!$ $\{x_i, \vec{r}\} \in CF(Y_N(K), s_m)$
and a <u>small</u> triangle $\tau_i \in \pi_2(\{x_i, \vec{r}\}, \Theta, \{x_0, \vec{r}\})$

satisfying $n_z(\tau) - n_w(\tau) = m - A(\{x_0, \vec{r}$







$j = m$

$\simeq C(\max\{i, j-1\} = 0)$

$\|$

$\widehat{CF}(S^3_N(K), s_0)$

Matt Hedden HFH 2/24/11

(3,4) torus knot

Exercises (A) Compute $CFK^{\infty}(S^3, T_{3,4})$

(Hint: It has a genus 1, doubly pted Heegaard diagram.

(B) Use surgery formula to compute

$$HF^{\pm}(S^3_N(T_{3,4}), s_i) \qquad \forall i \in \{0, 1, \ldots, N-1\}$$
$$HF^{\pm}(S^3_{-N}(T_{3,4}), s_i)$$

Rmk. • The proof of the surgery thm. used a triangle map that we typically associate to $W_N(K)$.

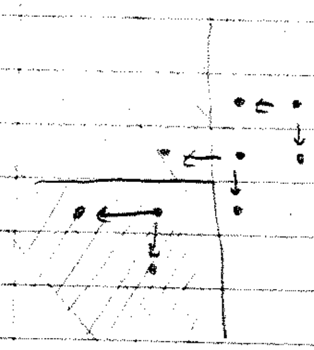
• N had to be large, because otherwise, we couldn't guarantee that $CF(Y_N(K), s_i)$ was generated by $\vec{x} = \{x_i, \vec{y}\} \subseteq$ winding region

$$\Phi_m : CF^-(Y_N(K), s_m) \xrightarrow[\text{is of form } L + H = \Phi_m]{\text{counts triangles, \&}} C(\max\{i, j-m\} < 0)$$

$$\downarrow \iota \qquad C(\max\{i,j-m\}<0) \hookrightarrow C(i<0)$$

$$\iota \circ \Phi_m : CF^-(Y_N(K), s_m) \to C(i<0) = CF^-(Y)$$

Forget about the j-filtration

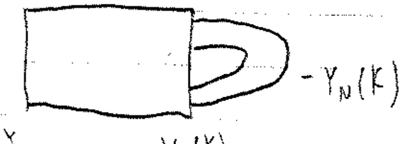


Prop. $\iota \circ \Phi_m \simeq f_{-W_N'(K), t_m}$,

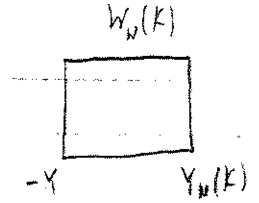
Chain homotopic

where $f_{-W_N'(K), t_m} : CF^-(Y_N(K), s_m) \to CF^-(Y, s)$

is the chain map on Floer homology complexes induced by



, t_m is, as usual, the unique $Spin^c$-structure on $-W_N'(K)$ s.t. $\langle c_1(t_m), [\hat{S}]\rangle + N = 2m$ & $t_m|_Y = s$

Given 2-handle cobordism 

the definition of $f_{W_N(K),t}(\vec{x}) = f_{W_{\alpha\beta\gamma}}(\vec{x} \otimes \Theta_{top})$

$\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$

$$= \sum_{\vec{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \ \sum_{\substack{\psi \in \pi_2(\vec{x}, \Theta, \vec{y}) \\ \mu(\psi) = 0 \\ s_w(\psi) = t}} \# \mathcal{M}(\psi) \cdot U^{n_w(\psi)} \cdot \vec{y}$$

<u>Rmk:</u>
- N in the theorem can be taken to be $\geq 2g(K) - 1$ (in the case that $Y = \mathbb{Q}HS^3$) (g = Seifert genus)
- We have an analogous formula for <u>all</u> N, but it's more complicated.

<u>Surgery Exact Triangle</u>

Suppose M is a 3-mnfld. w/ torus ∂, e.g. Y - nbhd(K).
Then we can Dehn fill along ∂M to obtain a closed 3-mnfld, $M_\lambda(K)$ (λ = choice of simple closed curve on ∂M.)

Suppose we have 3 curves in ∂M β, γ, δ s.t.
$\beta \cdot \gamma = -1$
$\gamma \cdot \delta = -1$
and $\delta \cdot \beta = -1$.

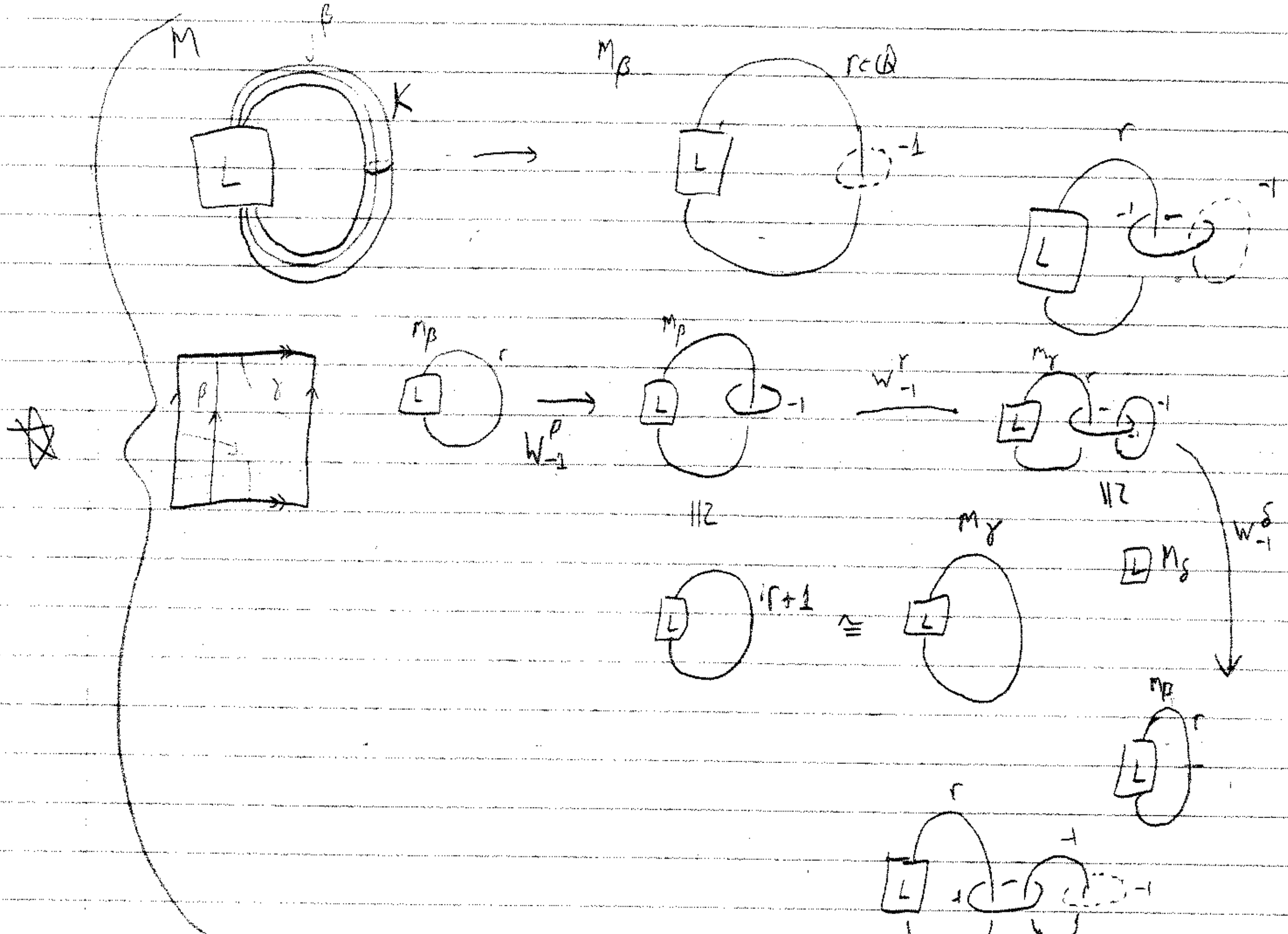
We say β, γ, δ form a <u>triad</u>.

<u>Thm.</u> Let M, β, γ, δ be as above.
Then $\exists$ long exact sequence

$$\cdots \to HF^\bullet(M_\beta) \longrightarrow HF^\bullet(M_\gamma) \longrightarrow HF^\bullet(M_\delta) \to \cdots$$

where $\bullet = \wedge, +$ (with ~~such~~ same sequences relating $-, \infty$ version if base ring is $\mathbb{Z}[[U]]$)

Note: For power series $-$, ∞, we don't need admissibility.



$$\cdots \longrightarrow HF^\circ(M_\beta) \longrightarrow HF^\circ(M_\gamma) \longrightarrow HF^\circ(M_\delta) \longrightarrow \cdots$$

$$\bigoplus_{t\in Spin^c} F^\circ_{W^\beta_{-1},t} \qquad \bigoplus F^\circ_{W^\gamma_{-1},t} \qquad \bigoplus F^\circ_{W^\delta_{-1},t}$$

Exercise: Show that if β, γ, δ are 3 slopes on a torus which form a triad, then the three 3-manifolds M_β, M_γ, M_δ obtained by Dehn filling along the slopes, are related as in ☆

Digression on Algebra (All mod 2)

Recall $A_1 \xrightarrow[\text{chain map}]{f_1} A_2 \rightsquigarrow$ Mapping cone complex $M(f_1)$

$$M(f_1) \cong (A_1 \oplus A_2, \partial)$$

$$\partial = \begin{bmatrix} \partial_1 & 0 \\ f_1 & \partial_2 \end{bmatrix}.$$

Mapping cones are natural wrt chain maps:

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "g_1"'] \arrow[dr, dashed, "H"] & A_2 \arrow[d, "g_2"] \\ B_1 \arrow[r, "f_2"] & B_2 \end{tikzcd} \qquad \begin{array}{c} g_2 \circ f_1 + g_1 \circ f_2 \\ = \\ H \circ \partial_1^A + \partial_2^B \circ H. \end{array}$$

$$\rightsquigarrow \quad \begin{array}{l} M(f_1) = (A_1 \oplus A_2, \partial) \\ \big\downarrow M(g_1 + g_2) \\ M(f_2) = (B_1 \oplus B_2, \partial) \end{array} = \begin{bmatrix} g_1 & 0 \\ H & g_2 \end{bmatrix}$$

$$0 \to A_2 \to M(f_1) \to A_1 \to 0$$

$$\Rightarrow$$

$$\begin{tikzcd} H_*(A_2) \arrow[r] & H(M(f_1)) \arrow[r] & H_*(A_1) \arrow[ll, bend left, "\delta = (f_1)_*"] \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & A_2 \arrow[r] \arrow[d, "g_2"] & M(f_1) \arrow[r] \arrow[d, "M(g_1, g_2)"] & A_1 \arrow[r] \arrow[d, "g_1"] & 0 \\ 0 \arrow[r] & B_2 \arrow[r] & M(f_2) \arrow[r] & B_1 \arrow[r] & 0 \end{tikzcd}$$

Def. $g : A \to B$ is a quasi-isomorphism if $g_* : H(A) \to H(B)$ is an isomorphism.

Lemma. Suppose we have

indices mod 3, obviously

$$\begin{tikzcd}
A_1 \arrow[rr, "f_1"] \arrow[dr, bend left=70, "H_1"] & & A_2 \arrow[dl, "f_2"] \arrow[ll, bend right=40, "H_2"'] \\
& A_3 \arrow[ul, "f_3"] \arrow[ur, bend right=70, "H_3"'] &
\end{tikzcd}$$

with (1) $f_{i+1} \circ f_i = \partial_{i+2} \circ H_i + H_i \circ \partial_i$

(2) $\gamma_i = f_{i+2} \circ H_i + H_{i+1} \circ f_i$ are quasi-isomorphisms.

Then, $M(f_i) \simeq A_{i-1} \quad \forall i = 1, 2, 3$

quasi-isomophic

In particular, $\exists$ l. e. s.

$$\begin{tikzcd}
\cdots \arrow[r] & H_*(A_{i+1}) \arrow[r, "(f_{i+1})_*"] & H_*(A_{i-1}) \arrow[r, "(f_{i-1})_*"] \arrow[d, phantom, "\cong" rotate=90] & H_*(A_i) \arrow[r, "(f_i)_*"] & \cdots \\
& & H_*(M(f_i)) & &
\end{tikzcd}$$

Pf.

$$M(f_i) \xrightarrow{\alpha_i} A_{i-1}$$

$$(A_i \oplus A_{i+1}, \partial) \longrightarrow A_{i-1}$$

$$\begin{bmatrix} H_i \\ f_{i+1} \end{bmatrix}$$

α_i is a chain map by Condition (1).

(Check. $\begin{bmatrix} H_i \\ f_{i+1} \end{bmatrix} \begin{bmatrix} \partial & 0 \\ f & \partial \end{bmatrix} = \begin{bmatrix} \partial H_i \\ \partial f_{i+1} \end{bmatrix}$.

$$A_{i-1} \xrightarrow{\beta_{i-1}} M(f_i)$$

$$\begin{bmatrix} f_{i-1} & H_{i-1} \end{bmatrix}$$ is a chain map.

Now, check $\alpha_i \circ \beta_{i-1} = \psi_i$, so by (2), this is q-isomorphism.

Opposite direction follows from same fancy diagram + Five Lemma.

<u>Exercise</u>. $\beta \circ \alpha$ is also quasi-isomorphism. $\square$

Matt Hedden HFH 3/1/11

Lemma Sps. A_i, $i=1,2,3$, $f_i : A_i \to A_{i+1}$, $H_i : A_i \to A_{i+2}$ (mod 3), are chain complexes, maps, homotopies s.t.

(1) $f_{i+1} \circ f_i = \partial H_i + H_i \circ \partial$

(2) $\sim = \quad \psi_i \quad H_{i+1} \circ f_i + f_{i+2} \circ H_i$ are quasi-isomorphisms.

Then, $M(f_i) \simeq A_{i-1}$.

Useful:

Thm. Sps. M is mnfld. with $\partial M \cong T^2$, and β, γ, δ are $\exists$ slopes forming a triad, then $\exists$ l.e.s.

$$\cdots \to HF^\circ(M(\beta)) \xrightarrow{F^\circ_{W(\beta)}} HF^\circ(M(\gamma)) \xrightarrow{F^\circ_{W(\gamma)}} HF^\circ(M(\delta)) \xrightarrow{F^\circ_{W(\delta)}} \cdots$$

Pf. (For hat version)

Consider the Heegaard multi-diagram

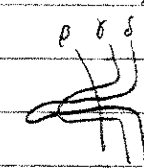
$$(\Sigma, \vec{\alpha}, \{\beta_1, \ldots, \beta_{g-1}, \beta_g\}, \{\gamma_1, \ldots, \gamma_g\}, \{\delta_1, \ldots, \delta_g\}, z)$$

where $(\Sigma, \vec{\alpha}, \{\beta_1, \ldots, \beta_{g-1}\})$ specifies M and $\beta_i \sim \gamma_i \sim \delta_i \quad \forall i = 1, \ldots, g-1$

$(\Sigma, \vec{\alpha}, \vec{\beta})$ specifies $M(\beta)$ small Hamiltonian perturbation

$(\Sigma, \vec{\alpha}, \vec{\gamma})$ $M(\gamma)$

$(\Sigma, \vec{\alpha}, \vec{\delta})$ $M(\delta)$.



Note $Y_{\beta\gamma} \cong \#^{g-1} S^1 \times S^2 \cong Y_{\gamma\delta}$ (β, γ, δ form triad $\Rightarrow$ adding curves β_g & γ_g, δ_g, connect sums on S^3)

$\cong Y_{\beta\delta}$

Define (as before)

$$f_1 \text{ by } f_{W(\beta)}(\vec{x}) = f_{\alpha\beta\gamma}(x \otimes \Theta_{\beta\gamma}) = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\psi \in \pi_2(x, \Theta, y) \\ \mu(\psi)=0 \\ n_z(\psi)=0}} \# \mathcal{M}(\psi) \cdot \vec{y}$$

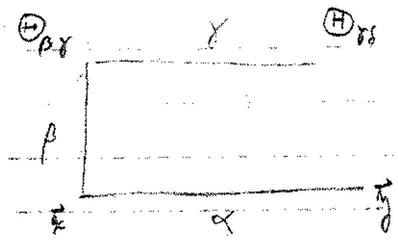
(f_2) by $f_{W(\gamma)}(\vec{x}) = f_{\alpha\gamma\delta}(x \otimes \Theta_{\gamma\delta})$

(f_3) by $f_{W(\delta)}(\vec{x}) = f_{\alpha\delta\beta'}(x \otimes \Theta_{\delta\beta'})$ small Hamiltonian perturbation

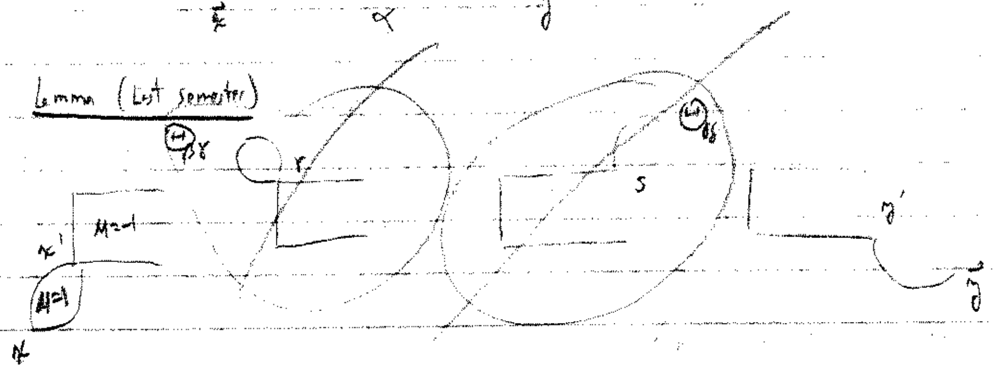
Need: $f_{i+1}\circ f_i \simeq \partial\circ H + H\circ\partial$

Define: $H_1(\vec{x}) := H_{\alpha\beta\gamma\delta}(\vec{x}\otimes\Theta_{\beta\gamma}\otimes\Theta_{\gamma\delta})$

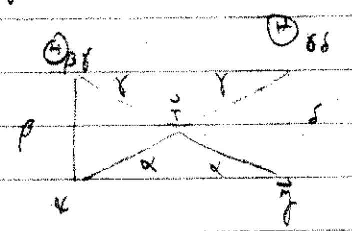
$$= \sum_{y\in \mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\substack{\square\in\pi_2(\vec{x},\Theta,\Theta,\vec{y}) \\ \mu(\square)=-1 \\ n_z(\square)=0}} \#\mathcal{M}(\square)\cdot\vec{y}$$



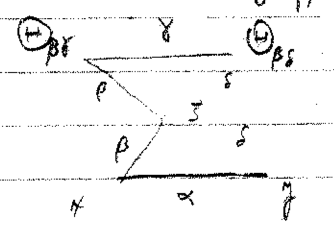
Lemma (Last semester)



$H\circ\partial$



$\partial\circ H$



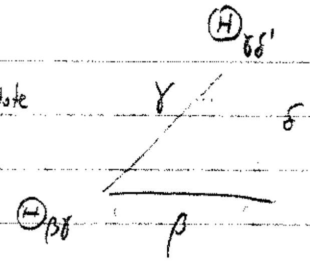
$f_{\alpha\gamma\delta}(f_{\alpha\beta\gamma}(x\otimes\Theta_{\beta\gamma})\otimes\Theta_{\gamma\delta})$ $\qquad$ $f_{\alpha\beta\delta}(x\otimes f_{\beta\gamma\delta}(\Theta_{\beta\gamma}\otimes\Theta_{\gamma\delta}))$

$\|$

$f_2\circ f_1(\vec{x})$

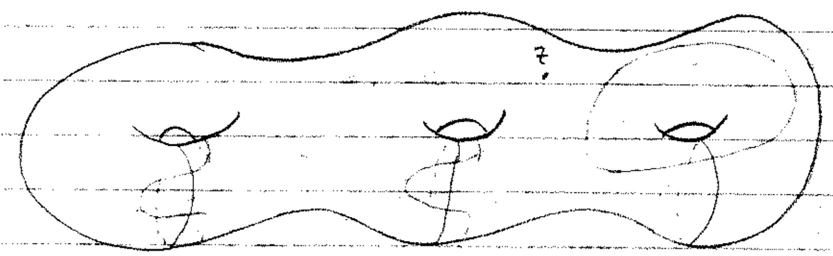
Claim: $f_{\beta\gamma\delta}(\Theta_{\beta\gamma}\otimes\Theta_{\gamma\delta}) = 0$

__Pf of Claim__ Note

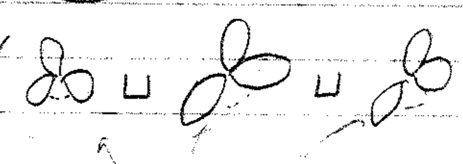


$Y_{\beta\gamma} = Y_{\gamma\delta} = Y_{\delta\beta} = \#^{g-1} S^1 \times S^2$

In fact, the Heegaard triple diagram specifying this situation is diffeomorphic to

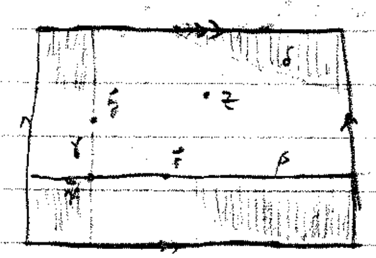


__Exercise.__ Show that $W_{\beta\gamma\delta} \cong \overline{\mathbb{CP}^2}$ - nbhd. (



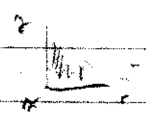
)

wedge of g-1 circles

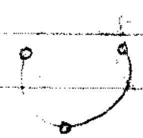


__Lemma__. For this genus 1 diagram, $\exists$ exactly 2 J-holomorphic triangles $\vec{x}, \vec{y}, \vec{r}$.

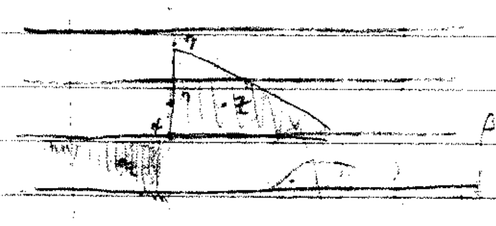
__Pf. of Lemma__. By RMT (w/ ∂) $\Rightarrow$



has a unique bi-holomorphism w/ standard triangle in $\mathbb{C}$:



Lift to Universal cover & Note similar lifts



It remains to verify that $\Upsilon_i = f_{i+2} \circ H_i + H_{i+1} \circ f_i$ are quasi-isomorphisms.
ETS for Υ_1 (as remaining cases follow by invariance of everything under cyclic permutation)

Uses triad condition.

$$f_3 \circ H_1(\vec{x}) = f_{\alpha\delta\beta'}\left(\underbrace{H_{\alpha\beta\gamma\delta}}_{B}\left(\vec{x} \otimes \Theta_{\beta\gamma} \otimes \Theta_{\gamma\delta}\right) \otimes \Theta_{\delta\beta'}\right) \quad \text{☆}$$

$$+ \qquad\qquad +$$

$$H_3 \circ f_1(\vec{x}) = \underbrace{H_{\alpha\gamma\delta\beta'}}_{A}\left(f_{\alpha\beta\gamma}\left(\vec{x} \otimes \Theta_{\beta\gamma}\right) \otimes \Theta_{\gamma\delta} \otimes \Theta_{\delta\beta'}\right) \quad Y_{\alpha\beta} \to Y_{\alpha\beta'}$$

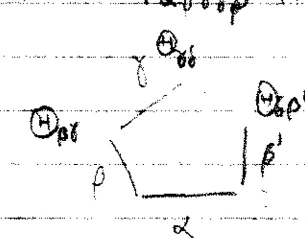
WTS: ☆ is a quasi-isomorphism.

Idea: We'll show that ☆ is chain homotopic to a quasi-isomorphism we encountered last semester, namely, $f_{\alpha\beta\beta'}$, the holomorphic triangle map we used to prove invariance of HF under isotopies (of β) that introduce 2 additional intersection points.

$$f_{\alpha\beta\beta'}\left(x \otimes \Theta_{top}\right) = x' + \text{higher order terms wrt Area filtration}$$

close pt.

$$\text{Consider} \quad P_{\alpha\beta\gamma\delta\beta'}\left(x \otimes \Theta \cdots \otimes \Theta\right) = \sum_{y \in T_\alpha \cap T_{\beta'}} \sum \# M(\pentagon) \cdot \vec{y}$$

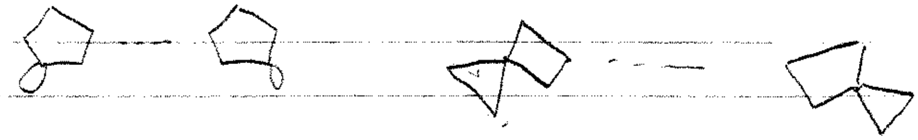


$\pentagon \in \pi_2(x, \Theta, \ldots, \Theta, y)$

$\mu(\pentagon) = -2$

$n_z(\pentagon) = 0$

Consider the ends of $M(\pentagon)$ with $\mu(\pentagon) = -1$.



A B, since vanish.

In the five terms contributing to the end of $\partial M(\pentagon)$ (not involving ∂P & $P\partial$)

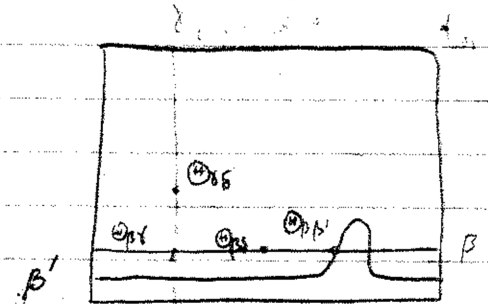
- Two vanish by previous claim.
- Two are identified with Υ_1
- Last is $f_{\alpha\beta\beta'}\left(x \otimes H_{\beta\gamma\delta\beta'}\left(\underbrace{\Theta_{\beta\gamma} \otimes \Theta_{\gamma\delta} \otimes \Theta_{\delta\beta'}}\right)\right)$

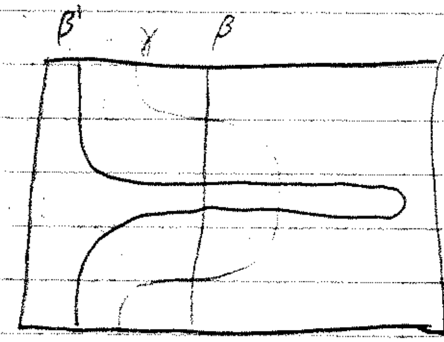
Thm. proved if we show that

Claim. $H_{\beta\gamma\delta\beta'}(\Theta \otimes \Theta \otimes \Theta) = \Theta_{\beta\beta'}$

Pf. Model calculation, as before.

Exercise: Analyze all holomorphic quadrilaterals in LHS connecting $\Theta_{\beta\gamma}$, $\Theta_{\gamma\delta}$, $\Theta_{\delta\beta'}$, $\Theta_{\beta\beta'}$ (should be exactly one).





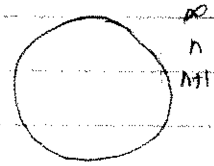
Exercise: Analyze the general case, i.e. include $\#^{g-1}$ copies of RHS

Matt Hedden HFH 3/3/11

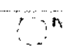
We've proved $\exists$ l.e.s.

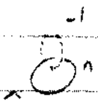
$$\to HF^\circ(M(\alpha)) \to HF^\circ(M(\beta)) \to HF^\circ(M(\gamma)) \to$$

Ex.



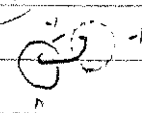
surgeries on the unknot form a triad.





$\widehat{F}_{W_n}$ $\widehat{F}_{W_{n+1}}$

$$\widehat{HF}(S^3 = M(\infty)) \to \widehat{HF}(L(n,1)) \to \widehat{HF}(L(n+1,1))$$



more precisely,

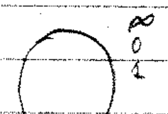
$$\widehat{HF}(S^3, s_0) \to \begin{array}{c} \widehat{HF}(L(n,1), s_{n-1}) \\ \oplus \\ \vdots \\ \oplus \\ \widehat{HF}(L(n,1), s_0) \end{array}$$

$$\downarrow$$

$$\begin{array}{c} \widehat{HF}(L(n+1,1), s_n) \\ \oplus \\ \vdots \\ \oplus \\ \widehat{HF}(L(n+1,1), s_0) \end{array}$$

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z}^n \longrightarrow \mathbb{Z}^{n+1}$$

Ex:



$$\widehat{HF}(S^3) \longrightarrow \widehat{HF}(S^1 \times S^2) \longrightarrow \widehat{HF}(S^3)$$

$$\mathbb{Z} \longrightarrow \begin{array}{c} \mathbb{Z}_i \\ \oplus \\ \mathbb{Z}_{i+1} \end{array} \longrightarrow \mathbb{Z}$$

$$\mathbb{Z} \to \begin{array}{c} \mathbb{Z}_i \\ \oplus \\ \mathbb{Z}_{i+1} \end{array} \nearrow \mathbb{Z} \quad \text{or} \quad \mathbb{Z} \searrow \begin{array}{c} \mathbb{Z}_i \\ \oplus \\ \mathbb{Z}_{i+1} \end{array} \to \mathbb{Z} \quad \text{or} \quad \mathbb{Z} \rightrightarrows \begin{array}{c} \mathbb{Z}_i \\ \oplus \\ \mathbb{Z}_{i+1} \end{array} \rightrightarrows \mathbb{Z} \quad ?$$

Absolute Gradings on Floer groups

Singular Homology : Axiom of Point - $H_*(pt.) \cong \begin{cases} \mathbb{Z} & *=0 \\ 0 & \text{else} \end{cases}$

We'll grade all of Floer homology by specifying that

$$\widehat{HF}_*(S^3) = \begin{cases} \mathbb{Z} & *=0 \\ 0 & \text{else}. \end{cases}$$

How to connect S^3 to another manifold?

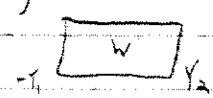
Given a cobordism from S^3 to Y, we'll define a grading of $HF(Y)$.
To a 4-mnfld. W with $\partial W = -S^3 \sqcup Y$, we have a map

$$HF^o_*(S^3, \mathfrak{t}|_{S^3}) \xrightarrow[F^o_{W,\mathfrak{t}}]{} HF^o_{*'}(Y, \mathfrak{t}|_Y)$$

The degree of this map is given by $\dfrac{c_1^2(\mathfrak{t}) - 2\chi(W) - 3\sigma(W)}{4}$ ☆

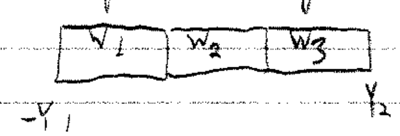
i.e. $gr\left(F^o_{W,\mathfrak{t}}(\xi)\right) - gr(\xi) = ☆$ for all ξ

The map F_W was constructed by



(1) Considering a self-indexing Morse function on W without 0 + 4 handles, with connected level sets.

(2)



(3) Define $F_{W,\mathfrak{t}} = F_{W_3} \circ F_{W_2} \circ F_{W_1}$

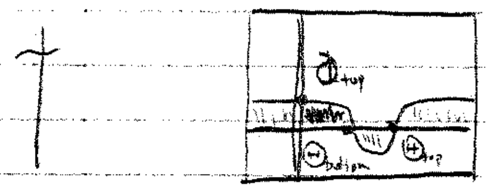
Note, $\partial W_1 \cong -Y_1 \sqcup Y_1 \overset{k}{\#} S^1 \times S^2$, for $k = \#$ index-1 critical pts

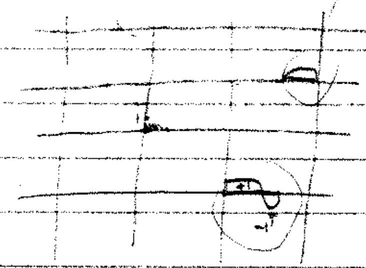
Given a H.D. for Y_1, a H.D. for $Y_1 \#^k S^1 \times S^2$ is obtained by

$$\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \xrightarrow{f_{W_1}} \vec{x} \otimes \Theta_{top}$$

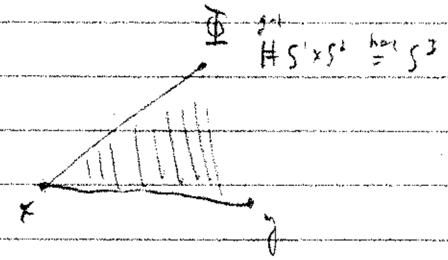
∂ condition not right.

F_{W_1} is the induced map on homology.





Multiplicities
$\Rightarrow$ ~~No~~ 1-handle rep.



$\sum F_{W_1,t}$

$\mathbb{Z}_{(0)} \longrightarrow \mathbb{Z}_{top}, \ \mathbb{Z}_{bottom}$

$$Spin^c(W_0(\text{unknot})) \cong Spin^c(S^1 \times S^2)$$
$$\text{Ⅲ}$$
$$H^2(W_0) \xrightarrow{i^*} H^2(S^1 \times S^2)$$

Note: Since Chern class of the $Spin^c$-structure associated to the triangle must agree with the Chern class of the restriction of this $Spin^c$-structure to the boundary, we have $c_1(t) = 0$, $c_1^2(t) = 0$.

1 1-handle, that's it.

So, $$gr\left(F^{0}_{W,t}\left(\Theta_{bottom}\right)\right) - gr(\vec{x}) = \frac{c_1^2 - 2\chi - 3\sigma}{4} = \frac{0 - 2(1) - 3(0)}{4} = -\frac{1}{2}$$

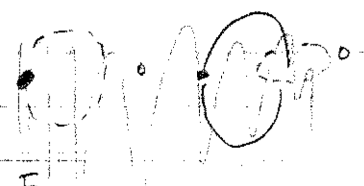
$gr(\vec{x}) = 0$ by Def. in S^3

$$\Rightarrow gr\left(F^{0}_{W,t}\left(\Theta_{bottom}\right)\right) = -\frac{1}{2}$$

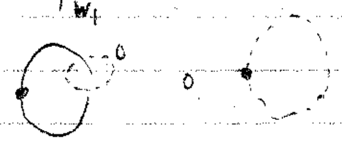
Why is the <u>definition</u> of the 1-handle map the right one?

$W =$  $\cong S^3 \times I = W$

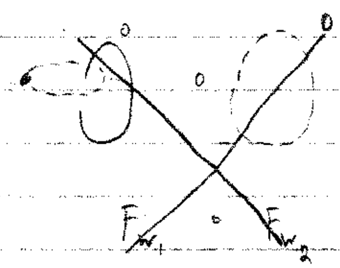
$\|$



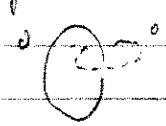
F_{W_1}



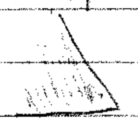
$$\underbrace{F_{W_2} \circ F_{W_1}}_{\text{Better be the identity!}} \Big(\underset{\substack{\| \\ \text{gen. of} \\ \widehat{HF}(S^3)}}{\vec{x}} \Big)$$



Interchanging the roles of β & γ in the picture $\uparrow$ (on previous page) is a triple diagram for



Θ_{top}



So

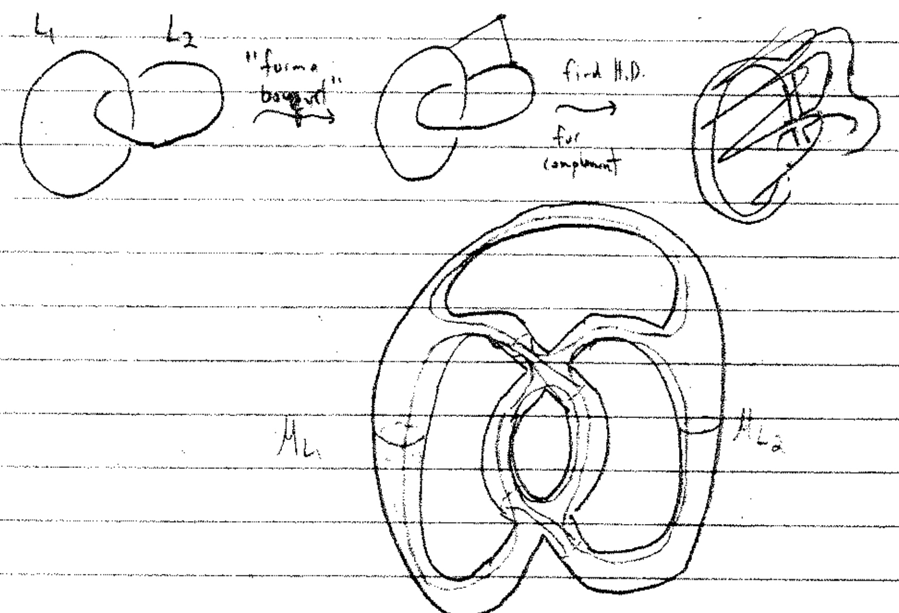
$$F_{W_2} \circ F_{W_1}(\vec{x}) \overset{\text{By Def.}}{=} F_{W_2}\left(\Theta_{top}\right) \overset{\text{By computation of holomorphic triangles.}}{=} \vec{x}$$

<u>Remark:</u> This is what we want the grading to be anyway, if this theory is to agree with Seiberg-Witten Theory.

Map $F_{W_3} : Y \#^k S^1 \times S^2 \to Y$ induced by 3-handles
$[x \otimes \Theta_{bottom}] \mapsto [x]$

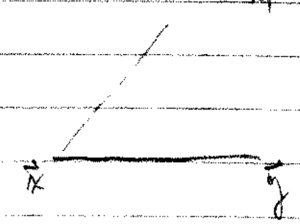
For 2-handles, we've seen how to define a map for a single 2-handle
(i.e. take a HD for $Y - K$ + form the triple diagram
$(\Sigma, \vec{\alpha}, (\beta_1, \ldots, \beta_{g-1}, \mu), (\gamma_1, \ldots, \gamma_{g-1}, \lambda), w)$,
and compute holomorphic triangles.

<u>Exercise</u>. Verify that $F(\overset{0}{\bigcirc} \cup \text{3-handle}) = Id$.



with properties
$(\Sigma, \{\alpha\}, \{\beta_1, \ldots, \beta_{g-|L|}, \mu_1, \ldots, \mu_{|L|}\}, \{\gamma_1, \ldots, \gamma_{g-|L|}, \lambda_1, \ldots, \lambda_{|L|}\}, w)$ with $\beta_1 \cong \gamma_1, \ldots, \beta_{g-L} \cong \gamma_{g-L}$

Define map counting 3-hdu $\Theta_{top} \in \#^{g-|L|} S^1 \times S^2$



So we've defined maps associated to cobordisms + asserted they shift degree by ★.

Concretely, we'd like to understand grading even when we don't have a nontrivial map.

One solution is to take a 2-handle cobordism $\boxed{W_2}$ over which $s \in Spin^c(Y)$

$-S^3 \quad Y.$

extends to some $t \in Spin^c(W_2)$

s extends to $t \iff \exists\, \psi \in \pi_2(x, \Theta, y)$ for $x \in \widehat{CF}(S^3, s_0)$

& some $y \in \widehat{CF}(Y^3, s)$.

Now, $$\deg(y) - \deg(x) = \underbrace{-\mu(\psi)}_{\substack{\text{Maslov index of triangle}}} + 2n_w(\psi) + \frac{c_1^2(s_w(\cancel{\psi})) - 2\chi(W) - 3\sigma(W)}{4}$$

t

(Maybe $(+)$)

Exists formula, analogous to Lipshitz' formula for $\mu(\phi)$ in terms of domain (Sarkar: Maslov index of triangles)