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Call the homotopy class of solutions $\phi_{\alpha}$, $\phi_{\beta}$.

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Call the homotopy class of solutions $\phi_{\alpha}$, $\phi_{\beta}$.

Homotopy class $\phi_{\alpha}$
Finally, we have \( \pi_2(\mathbb{R}^2, \mathbb{R}) \times \pi_2(\text{Sym}^d(\mathbb{R}^2), \mathbb{R}) \rightarrow \pi_2(\mathbb{R}^2, \mathbb{R}) \)

\[ \phi \times S \rightarrow \phi \ast S \]

Exercise: show that \( \pi_2(\mathbb{R}^2, \mathbb{R}) \) is a group.

and \( \pi_2(\mathbb{R}^2, \mathbb{R}) \) (or \( \pi_2(\mathbb{R}^2, \mathbb{R}) \)) acts on \( \pi_2(\mathbb{R}^2, \mathbb{R}) \) from the left (right).

Proposition 3: function \( \mu : \pi_2(\mathbb{R}^2, \mathbb{R}) \rightarrow \mathbb{Z} \) called the Maslov index satisfying:

1. Additivity \( \mu(\phi \ast \phi_2) = \mu(\phi) + \mu(\phi_2) \), \( \phi, \phi_2 \in \pi_2(\mathbb{R}^2, \mathbb{R}) \), \( \phi_2 \in \pi_2(\mathbb{R}^2, \mathbb{R}) \)

2. Inverses \( \mu(\phi^{-1}) = -\mu(\phi) \)

3. Sphere Additivity \( \mu(\phi \ast S) = \mu(\phi) + 2\langle c_1(J), H_2(S) \rangle \)

\( H_2 : \pi_2(\text{Sym}^d(\mathbb{R}^2)) \rightarrow H_2(\text{Sym}^d(\mathbb{R}^2)) \)

is the Hirzebruch map.

and \( c_1 \) is the Chern class.

From (3), it follows that

4. (constant) Suppose \( \phi \in \pi_2(\mathbb{R}^2, \mathbb{R}) \) is constant map. Then \( \mu(\phi) = 0 \).

The Maslov index is the expected dimension of the space \( \mathcal{M}(\phi) \) of \( J \)-holomorphic disks.

\[ \mathcal{M}(\phi) = \{ u : [0,1] \times \mathbb{R} \rightarrow \text{Sym}^d(\mathbb{R}^2) \mid u(1) = \phi, \text{ unit norming class of approximate disks} \} \]

\[ du \circ i = J du \]

This is the space of \( J \)-holomorphic disks composing to \( \phi \) in homotopy class \( \phi \).

\[ \mathcal{M}(\phi) = \mathcal{M}(\phi)/\mathcal{R} \]

is a space of up to isometry \( J \)-holo...

which should be thinking is figure out what there

\[ \overline{\mathcal{M}} : B \rightarrow \mathbb{L} \]

\[ \overline{\mathcal{M}}(u) = du \circ i - J du \]

Problem: \( u \rightarrow \overline{\mathcal{M}}(u) \)

\( J \)-holomorphic operators

Wants \( 0 \) to be a regular value for this map, so that

\[ \overline{\mathcal{M}}^{-1}(0) \] will be a smooth manifold.

(0, Implicit-Function-Theo in some infinite diml. setting)
We can check for regular values at the level of derivatives:

\[ D_u \mathcal{J} : T_u \mathcal{B} \to T_{\mathcal{J}(u)} \mathcal{L} \]

Want this to be surjective \( \forall u \in \mathcal{J}^{-1}(0) \).

If we can do this, then \( \mathcal{M}(0) \) is a manifold of dimension \( \text{dim}(\ker(D_u \mathcal{J})) \).

Define the index \( (D_u \mathcal{J}) = \text{dim}(\ker(D_u \mathcal{J})) - \text{dim}(\text{coker}(D_u \mathcal{J})) \).

Want \( \text{dim}(\text{coker}(D_u \mathcal{J})) = 0 \), in which case \( \mathcal{M}(0) \) is a manifold of dim. \( \text{index}(D_u \mathcal{J}) \).

For us, Maslov index = \( \text{index}(D_u \mathcal{J}) \).

Define \( \partial : C(\Sigma, \mathcal{B}, \mathcal{P}) \to C(\Sigma, \mathcal{B}, \mathcal{P}) \) by

\[ \partial \mathcal{X} = \sum_{\gamma \in \mathcal{X}} \sum_{\phi \epsilon \text{Maslov}(\Phi)} \mu(\phi) \cdot \mathcal{F}_\phi \cdot \mathcal{P} \]

\[ \mu(\phi) = 1 \]

Issues

- Ensure \( \mathcal{M}(0) \) is smooth and existed derivative equal \( \mathcal{M}(0) \). (Achieving transversality)

  We do this usually by varying the almost complex structure \( J : T_{\text{SU}(2)} \mathcal{B} \to T_{\text{SU}(2)} \mathcal{B} \)

- Is the count finite? (Compactness)

- Is \( \partial^2 = 0 \) ?

Thm. (9.5) We can achieve transversality and compactness so that \( \partial \) is well-defined

and \( \partial^2 = 0 \). Moreover, \( H^*_c(\Sigma, \mathcal{B}, \mathcal{P}, \partial) = HF^\infty(\gamma) \) is

independent of \( (\Sigma, \mathcal{B}, \mathcal{P}) \), depending only on \( \Gamma \) smoothly.

But, for \( Y \in \mathcal{B} \), \( H_c^*(S^3, \mathcal{B}) = H_c(Y, \mathcal{B}) \),

\[ \text{rank}(HF^\infty(\gamma)) = |H_1(Y; \mathcal{B})| \]

Idea: Let's refine this moment by introducing a basepoint.

Def. A pointed Heegaard Diagram \( (PHD) \) is a HD \( (\Sigma, \mathcal{B}, \mathcal{P}) \) with a distinguished basepoint \( \mathcal{P} \in \mathcal{P} \).

\[ (\Sigma, \mathcal{B}, \mathcal{P}, \mathcal{P}) \]
The basepoint produces a codimension 2 submanifold of $\text{Sym}^{r}(\mathbb{C}^2)$, denoted $V^r = \{ z \mid z \in \text{Sym}^{r}(\mathbb{C}^2) \}$. Let $V^r_{\mathbb{Q}}$ be the image of $V^r$ under a morphism $\Phi: \mathbb{C}^2 \to \mathbb{Q}^2$. Consider $\# \text{Im}(\Phi) \cap V^r_{\mathbb{Q}}$. This depends only on $[\Phi] = \phi$.

Let $\# \text{Im}(\Phi) \cap V^r_{\mathbb{Q}} = \# \text{Im}(\Phi) \cap V^r_{\mathbb{Q}}^\perp$. Then $[\Phi] = \phi$.

The $4$ different answers that are obtained in different views:

- $C_4(y) = \bigoplus \mathbb{Q}/2 \langle x \rangle$
- $C_{4+}(y) = \bigoplus \mathbb{Z}_2 \langle x \rangle$
- $C_{4-}(y) = \bigoplus \mathbb{Z}_2 \langle u, u^{-1} \rangle$
- $C_{4^+}(y) = \bigoplus \left( \mathbb{Z}_2 \langle u, u^{-1} \rangle / \langle u \rangle \right)$

Let $\Phi : \mathbb{Q}_p \to \mathbb{Q}_p$ and $\Phi : \mathbb{Q}_p(\sqrt{2}) \to \mathbb{Q}_p(\sqrt{2})$.

$$\phi = \sum_{\Phi \in \mathbb{Q}_p} \sum_{\Phi \in \mathbb{Q}_p(\sqrt{2})} \#	ext{Im}(\Phi) \cdot \hat{\Phi}$$

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Let $\mathcal{C} = \mathbb{Z}[\pi, (\gamma, \delta)]$. In general, we will take kernels of twisted coefficients. Let $\mathcal{C}(X; M)$, $\mathcal{M}$ modulo over $\mathbb{Z}[\pi, (\gamma, \delta)]$.

See Dave's 1994 notes, Algebraic Topology.
\[ \begin{align*}
\mathcal{J}(\phi) & \subset \pi_2(x,y) \\
\mu(\phi) & \in \mathbb{Z} \quad \text{monodromy} \\
\text{and: H0} & \quad (\mathbb{Z} \subset \mathbb{R}, \mathbb{Z}) \\
n_{\mathbb{Z}}(\phi) & = \# V_{\mathbb{Z}} \cap \text{Im } \mathcal{U}, \quad \text{Im } \mathcal{U} = \phi \in \pi_2(x,y) \\
\text{(Positivity Principle)} \\
\Leftrightarrow & \\
\mathbb{Z} \quad \mathbb{C}^F & \quad \text{requires } u + \mathcal{J}(\phi) \text{ to miss } V_{\mathbb{Z}}, \text{ i.e. } n_{\mathbb{Z}}(\phi) = 0 \\
\mathbb{Z} \quad \mathbb{C}^{-F} & \quad \text{"allows" } n_{\mathbb{Z}}(\phi) \text{ to be arbitrary, but keep track by } \bigcup n_{\mathbb{Z}}(\phi) \\
\mathbb{Z} \quad \mathbb{C}^{-F} & \quad \text{Lemma: If } \mathcal{J}(\phi) \neq \emptyset, \text{ then } n_{\mathbb{Z}}(\phi) \geq 0. \\
\mathbb{C} \text{ Hull: Sym}^F(\mathbb{Z}) & \quad \text{a complex manifold with complex structure } \text{Sym}^F(\mathbb{Z}), \\
\text{where } & \quad \mathbb{Z} \text{ is a complex structure on } \mathbb{Z}. \\
\text{Now, } & \quad (\mathbb{Z} = \text{Sym}^F(\mathbb{Z})) \text{ : holomorphic map } [\phi, \mathbb{Z}] \text{ on } \mathbb{C} \rightarrow \text{Sym}^F(\mathbb{Z}) \\
\text{is a holomorphic map}, \quad & \quad w \in \mathcal{J}(\phi) \text{ gives a holomorphic map } \mathbb{D} \rightarrow \text{Sym}^F(\mathbb{Z}) \\
p_{\mathbb{D}}(\phi) & \subset \text{Im } u \cap V_{\mathbb{Z}}, \text{ but Im}(\mathcal{U}) \text{ and } V_{\mathbb{Z}} \text{ are complex submanifolds of complementary dimension. } \\
\text{Exercise: Holomorphic submanifolds of a complex manifold intersect non-transversally, hence transverse.} \\
\text{Condition: } & \quad (\mathbb{D} \in \mathbb{U} \cap T_p) \Rightarrow \mathbb{D} \notin V_{\mathbb{Z}}. \\
\Rightarrow & \quad n_{\mathbb{Z}} \geq 0. \\
\text{(Now, this is a homotopy-theoretic condition.) So, perturbing the complex structure (as in almost)} \\
\text{complex structure) preserves } n_{\mathbb{Z}} \geq 0. \\
\text{Alternatively, pick an almost-complex structure } \mathbb{J} \in \text{Sym}^F(\mathbb{Z}) \text{ which agrees with } \\
\text{Sym}^F(\mathbb{Z}) \text{ in } \text{Im}(V_{\mathbb{Z}}). \\
\text{Then, the calculation is the same.}
\end{align*} \]
Lemma: \exists a short exact sequence of chain complexes
\[0 \to CF^- \xrightarrow{i} CF^\infty \xrightarrow{f} CF^+ \to 0\]

coming from the theorem.

\[0 \to \mathbb{Z}_2[U] \to \mathbb{Z}_2[U, U^n] \to \mathbb{Z}_2[U, U^n]/\mathbb{Z}_2[U] \to 0\]

Remark: The main content of this lemma is that \(i, \rho\) are chain maps.

\[i \circ \rho(x) = \rho \circ \rho(x) = \rho \circ \rho(x)
\]

Proof: \(\alpha = \{x, y\} + 1 \cdot U \cdot y\)

\[\alpha = \alpha \cdot \alpha = \alpha \cdot \alpha = \alpha \cdot \alpha
\]

\[\mathbb{Z}_2 \mathbb{Z} \oplus \mathbb{Z}_2 \mathbb{Z} \oplus \mathbb{Z}_2 \mathbb{Z} \oplus \mathbb{Z}_2 \mathbb{Z} \oplus \mathbb{Z}_2 \mathbb{Z} \oplus \mathbb{Z}_2 \mathbb{Z}
\]

If \(n_2(x) < 0\), then \(\phi\) with \(\rho(x) \neq 0\), set \(\phi = \pi_2(x, y)
\]

Then \(U \cdot \phi\), \(\phi \in \partial x\) would not be defined.

By proceeding lemma ensures \(n_2(x) \neq 0\), \(\phi\) with \(\rho(x) \neq 0\).

\[\partial^- : CF^- \to CF^-, \quad b^+ : \partial^+ \cdot \partial^+ = \partial^+ \cdot \partial^+ = \partial^+, \quad \text{at some different point.}
\]

Argument above:

\[\partial^- : CF^- \to CF^-, \quad \partial^+ : CF^+ \to CF^+
\]

Any time \(\omega\) has a subsequence of chain maps \(\omega\), then \(\omega\) - s.e.s.

\[0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} B/A \to 0
\]

Exercise: \(0 \to \ker U \to CF^+ \xrightarrow{U} CF^4 \to 0\)

\[\hat{CF} \xrightarrow{\beta} \ker U \cong \text{a chain map}
\]

(4) Show that \(\hat{CF} \cong \ker U\).
What I understand $\text{CF}^+$ in terms of $\text{HD}$.

Example: Consider $L(3,1)$ or $L(-3,1)

\text{L}^+ = \mathbb{Z}^2 \cup \mathbb{Z} \cup \mathbb{Z} \cup \mathbb{Z}^*$.

$\text{CF}^+(L(3,1)) = \text{CF}^+(x_1^+) + \text{CF}^+(x_2^+) + \text{CF}^+(x_3^+)$

$\pi_2(x_1, x_2) = \pi_2(x_2, x_3) = \pi_2(x_3, x_1) = \emptyset$.

Find $\pi_2(x_1, x_2) = \emptyset$.

Path from $x_1$ to $x_2$ along $\alpha$ followed by a path from $x_2$ to $x_3$ along $\beta$.

Similarly, $\pi_2(x_2, x_3) = \pi_2(x_3, x_1) = \emptyset$.

$\pi_2(x_1, x_1) = \text{constant}$.

$\pi_2(T^2) \to \pi_2(T^2, x \cup \rho) \to \pi_1(\alpha \cup \rho) \to \pi_1(T^2)$

$\partial: \text{CF}^+(L(3,1)) \to \text{CF}^+(L(3,1))$ is trivial.

$\partial^2 = 0$, so $\text{CF}^+ \cong HP^+$

Similarly, $\text{CF}^- \cong HP^-$.

$0 = 1, -1, \infty$. 

Mistake at end of Last Time:

Calculation of $\pi_n (\mathbb{C}, \mathbb{C})$ for $\mathbb{C}^2$:

$$\pi_2 (2 \pi \beta) \rightarrow \pi_2 (\mathbb{C}^2) \rightarrow \pi_2 (\mathbb{C}^2, \mathbb{C}^2, \mathbb{C}^2) \rightarrow \pi_2 (\mathbb{C}^2, \mathbb{C}^2, \mathbb{C}^2)$$

$$\Omega (S^3 \times \mathbb{C}) \rightarrow \mathfrak{X} (\Pi_2, \Pi_3) \rightarrow \mathfrak{X} (\Pi_2, \Pi_3)$$

$$\Pi_2 \times \Pi_3 \rightarrow \mathfrak{X} (\Pi_2, \Pi_3) \rightarrow \mathfrak{X} (\Pi_2, \Pi_3)$$

For $\pi_2 (\mathbb{C}^2, \mathbb{C}^2) = \pi_2 (\mathbb{C}^2) = 0$, we have

$$0 \rightarrow \mathfrak{X} (\pi_2, \pi_2) \rightarrow \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$$

Therefore, $\pi_2 (\mathbb{C}^2, \mathbb{C}^2) = 0$.

(1) How to tell if $\pi_2 (\mathbb{C}, \mathbb{C}) \neq 0$, and, if so, how many $\phi \in \pi_2 (\mathbb{C}, \mathbb{C})$? (i.e., what is $\pi_2 (\mathbb{C}, \mathbb{C})$?)

(2) What are $\mathbb{Z} \in \Pi_2 \cap \Pi_3$?
Exercise 7.24: Show that \((\Sigma, (\alpha_1, \alpha_2), \beta_1)\) is a Heegaard diagram for the complement of a ribbon of the trefoil knot.

(i) Show that \((\Sigma, (\alpha_1, \alpha_2), (\beta_1, \beta_2))\) is a Heegaard diagram for \(N\) - surgery on the trefoil knot, and determine \(n\).

Let \(\alpha \in \pi_1 \cap \pi_2\), where \(\alpha\) is the boundary of an intersection points between \(\alpha = \beta\) curves such that \(\alpha \cap \beta\) common and parallel axes.

\[
\alpha_2 \cap \beta_2 = \{ \kappa_1, \ldots, \kappa_k \}
\]

\[
\alpha_1 \cap \beta_2 = \{ \mu_1, \ldots, \mu_k \}
\]

\[
\alpha_1 \cap \beta_1 = \{ m_1, m_2 \}
\]

\[
\alpha_1 \cap \beta_1 = \{ n_1, \ldots, n_k \}
\]

(3) Calculate \(H_1(M, \mathbb{Z})\) specified by \(H_0 = \mathbb{Z}\).

Calculate \(H_1\left(S^3 \ominus (K), \mathbb{Z}\right) \cong \mathbb{Z} / n\mathbb{Z}\) for \(n = 0, 1, 2, 3\).

\[
\pi_2(x, \phi)^2 \rightarrow \pi_1 \rightarrow \pi_0 \rightarrow \mathbb{Z}
\]

For \(\pi_2(x, \phi) \neq 0, \phi = \phi_1, \phi_2, \exists\)

\[
\left( F^2, F_0, \bar{F}, \bar{F}_0, \phi \right) \rightarrow \left( Z \rightarrow Z, \lambda_{\phi}, \lambda_{\bar{F}}, \lambda_{\bar{F}_0} \right)
\]

\[
(\bar{F}^2, a, b, c, \eta) \rightarrow u^\phi(\sigma) (\text{full-twist}) \rightarrow \text{boundary of full covering map}
\]

\[
\Phi = \phi_1 \circ \Phi \rightarrow u : (D^2, e_0, e_1, i) \rightarrow (S^3 \ominus (Z), \pi_2, \omega, \eta)
\]
Notice that \( \phi \mid_{\mathbb{T}_a \times \mathbb{R}} : \mathbb{T}_a \times \mathbb{R} \to \mathbb{T}_a \) (resp. \( \rho \)) is an actual (non-branched) covering map.

\( \phi \) (this pulls back)

When restricted to \( (\mathbb{C}_a, \mathbb{C}_b, \mathbb{C}_c, \mathbb{C}_d) \), \( \phi^* : \mathbb{C} \to \mathbb{C} \) is a \( g \)-fold covering map.

The diagram leads to

\[
\begin{array}{c}
\{ \alpha : \mathbb{D}^2 \to \mathbb{C} \} \\
\downarrow \text{in } \mathcal{S}_{\mathbb{T}} \end{array}
\begin{array}{c}
\{ \phi^2 \to \mathbb{Z} \} \\
\downarrow \text{in } \mathcal{S}_{\mathbb{T}}
\end{array}
\]

use in \( \mathcal{S}_{\mathbb{T}} \).

For \( (\mathbb{R}^2, a, b, \alpha, \beta) \),

\[
\begin{array}{c}
\mathbb{R}^2 \to \mathbb{C} \\
\alpha \end{array}
\]

is \( g \)-branched \( g \)-fold covering which is

\[
(\mathbb{R}^2, a, b, \alpha, \beta, \cdots)
\]

a covering on \( a, b, \alpha, \beta, \cdots \).

In particular, \( a \) is just \( g \) disjoint intervals.

\[
\begin{array}{c}
\{ \mathbb{R} \} \\
\downarrow \text{in } \mathcal{S}_{\mathbb{T}}
\end{array}
\begin{array}{c}
\{ \mathbb{R} \} \\
\downarrow \text{in } \mathcal{S}_{\mathbb{T}}
\end{array}
\]

\( \mathbb{R} \) is a surface with boundary, with its boundary divided into \( 2g \) arcs,

meeting along \( 2g \) pts., with \( g \) arc labeled \( a \).

\[ F^2 \to \mathbb{C} \] such \( a \to g \) arcs and \( g \) \( \times \) curves

\[ b \]

hence \( a \) to the \( g \)-tuple of pts. \( \xi \) such \( \xi \in \mathcal{S}_{\mathbb{T}} \).

Conversely,

\[ \mathcal{S}_{\mathbb{T}} \text{ we have } \begin{array}{c}
\mathbb{R}^2 \to \mathbb{C} \\
\phi \end{array} \text{ satisfying all the conditions (1) } \phi \text{ is } g \text{-branched } \rightarrow \text{ horizontal } \text{ and } \beta \\
(2) \text{ sends } a, b, \cdots \text{ in } \psi
\]

\( \Rightarrow \) a Whitney disk \( w : \mathbb{R}^2 \to \mathcal{S}_{\mathbb{T}} \) connecting \( \alpha \) to \( \beta \).

To see this define \( \varphi \) s on \( \partial \mathbb{R}^2 \); \( \varphi(a) - \varphi(b) \).
\[
\pi_2(\mathbb{R}^3, \emptyset) \cong \text{Homotopy class of maps } \mathbb{R}^2 \to \mathbb{R}^3
\]

where \( \mathbb{R}^3 \) is a branched cover \( \phi : \mathbb{R}^3 \to \mathbb{R}^3 \) over \((a, b, \kappa, \eta)\).

\[
\pi_2(\mathbb{R}, \emptyset) \cong \pi_2(\mathbb{R}, \emptyset)
\]

For \( \emptyset \), \( \pi_2(\mathbb{R}, \emptyset) \neq \emptyset \). \( \Rightarrow \exists \mathbb{R}^2 \to \mathbb{R}^3 \)

Consider \( \Phi \mid_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^3 \)

\( \Phi \mid_{\mathbb{R}^2} \subset \mathbb{R} \sqcup \mathbb{R}^2 \) as a collection of sub-\( \mathbb{R}^2 \)s connecting \( U \) and \( \mathbb{R}^2 \).

\( \Phi \mid_{\mathbb{R}^2} \subset \mathbb{R} \sqcup \mathbb{R}^2 \) as a collection of sub-\( \mathbb{R}^2 \)s connecting \( U \) and \( \mathbb{R}^2 \).

\( \Phi \mid_{\mathbb{R}^2} \subset \mathbb{R} \sqcup \mathbb{R}^2 \) as a collection of sub-\( \mathbb{R}^2 \)s connecting \( U \) and \( \mathbb{R}^2 \).
Ex.

\[ \partial F^2 \rightarrow \gamma, \eta \quad (i.e. \pi_2(\eta, \eta) \neq \emptyset) \]

Then \[ \partial F^2 \rightarrow \gamma' \cup \eta' \quad \text{collection of oriented arcs connecting } \gamma \leftrightarrow \eta \]

\( \text{clay } \gamma \leftrightarrow \text{clay } \eta, \text{ and } \gamma \leftrightarrow \eta \text{ clay } \gamma' \leftrightarrow \text{clay } \eta' \text{ resp.} \)

Prop. If \( [\gamma', \eta'] = 0 \in H_1(\Sigma) \), then \( \gamma' \cup \eta' = \partial \Sigma^2(\mathbb{P}^2) \).

But, we have to account for all possible paths \( \gamma \leftrightarrow \eta \).

Any other \( \gamma' \leftrightarrow \eta' \) differs from \( \gamma' \cup \eta' \) by the addition/subtraction of some \( \omega \leftrightarrow \omega' \).

\[ \omega \leftrightarrow \omega' \rightarrow \omega_1 \circ \omega_2 \rightarrow \omega_0 \]

Prop. 3. \( [\gamma_0 \cup \gamma_0'] = 0 \in H_1(\Sigma) \)

This is precisely the part of whether any path could be \( \partial F^2 \) is null-homologous.

Def. \( E(\gamma, \eta) := \langle \gamma \cup \eta \rangle \in H_1(\Sigma) \)

for any \( \gamma \leftrightarrow \eta \) connecting \( \gamma \leftrightarrow \eta \) along \( \alpha \), \( \gamma \rightarrow \gamma' \) along \( \gamma \).

Prop. If \( \pi_2(\alpha, \eta) \neq \emptyset \), then \( E(\gamma, \eta) = 0 \).

Prop. If \( E(\gamma, \eta) = 0 \), then \( \pi_2(\alpha, \eta) \neq \emptyset \).

Prop. If \( E(\gamma, \eta) = 0 \), then \( \gamma_0 \cup \gamma_0' \) which is null-homologous in \( \Sigma \).

\[ \Rightarrow \exists F^2 \rightarrow \Sigma \quad \text{satisfying the boundary conditions.} \]

This suffices to show that \( F^2 \) admits a branched covering \( F^2 \rightarrow \mathbb{P}^2 \).

\[ \text{This is true, but we will not prove it. Exercise} \]
$\mathcal{S}_n \ast \tau_2 : (\tilde{x}, \tilde{y}) \neq \emptyset \iff \mathcal{E}(\tilde{x}, \tilde{y}) = 0$

Nov, 19. If $\tau_2 (\tilde{x}, \tilde{y}) \neq \emptyset$, how many elements does it have?

Prompt: Exercises: what were asking is how many maps

$[-2, \frac{1}{4}] \rightarrow \mathbb{R}^2$?

$\Phi : (F^a, a, b, \nu, \gamma) \rightarrow (\bar{\mathbb{Z}}, \bar{\mathbb{R}})$

$[F^a, a \cup b] \in H_2 (\bar{\mathbb{Z}}, \bar{\mathbb{R}})$

\[ \begin{array}{ccc}
\cdots & \cdots & \cdots \\
\begin{array}{c}
\text{Diagram}
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\text{Diagram}
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\text{Diagram}
\end{array}
\end{array} \]

$H_0 (\mathbb{P}, \mathbb{R}) \rightarrow H_2 (\mathbb{Z}) \rightarrow H_2 (\mathbb{Z}, \mathbb{R} \cup \mathbb{R}) \rightarrow H_1 (\mathbb{Z}, \mathbb{R} \cup \mathbb{R}) \rightarrow H_0 (\mathbb{Z}, \mathbb{R} \cup \mathbb{R})$

\[ \begin{array}{ccc}
\mathbb{Z} & \mathbb{Z} & \mathbb{Z}
\end{array} \]

What if $\text{im}(i) = \ker(j)$?

$0 \rightarrow \mathbb{Z} \rightarrow H_2 (\mathbb{Z}, \mathbb{R} \cup \mathbb{R}) \rightarrow \ker j$

For $g \geq 1$:

$x \in \ker (\delta) \iff \sum_{i=1}^{g} \mathcal{V}(\mathcal{W}) = 0 \in H_1 (\mathbb{Z})$

Proof. $\tau_2 (\tilde{x}, \tilde{y}) = \mathbb{Z} \oplus \ker (\text{Span } [\tilde{x}] + \text{Span } [\tilde{y}] \rightarrow H_1 (\mathbb{Z}))$, if $\tau_2 (\tilde{x}, \tilde{y}) \neq \emptyset$.

Recall: For homology, $\tau_2 (\tilde{x}, \tilde{y})$ is constant. The trick is that in this case ($g=1$), we can take a $\mathcal{V}$-construction is easily only for disks.

How to interpret $\ker (\text{Span } [\tilde{x}] + \text{Span } [\tilde{y}] \rightarrow H_1 (\mathbb{Z})) = H_1 (\mathbb{Z}) / (\text{Span } [\tilde{x}] + \text{Span } [\tilde{y}])$

Proof. $H_1 (\mathbb{Z}) / (\text{Span } [\tilde{x}] + \text{Span } [\tilde{y}]) \rightarrow H_1 (\mathbb{Z}, \mathbb{Z}) \cong H^1 (\mathbb{Z}, \mathbb{Z})$

$\ker \delta \cong H_2 (\mathbb{Z}, \mathbb{Z})$
Proof. Mayer-Vietoris for the Hurewicz diagram:

\[ Y = (\Sigma \times I) \cup \{ \alpha \text{-handlebody} \} \cup \{ \beta \text{-handlebody} \} \]

\[ H_2 = H_\beta \]

\[ H_1(\Sigma) \rightarrow H_1(\Sigma_\alpha) \oplus H_1(\Sigma_\beta) \rightarrow H_1(Y) \rightarrow 0 \]

\[ H_2(Y) \leftarrow H_2(\Sigma_\alpha) \oplus H_2(\Sigma_\beta) \leftarrow H_2(\Sigma) \leftarrow H_3(Y) \]

\[ H_2(Y) \cong \ker i \cong \ker \left( \Sigma \rightarrow \Sigma_\alpha \oplus \Sigma_\beta \rightarrow H_1(\Sigma) \right) \]

Some notes:

- Collected for future use
- Visit for all homotopies
- Lightbox handlebodies