HFH   Matt Hedden   4/7/11

Last Time:

- Finished discussion of the Khovanov $\longrightarrow \widehat{HF}(\Sigma(L))$ spectral sequence,
  + mentioned its natural generalization to arbitrary surgery diagrams.
- ~~Open~~ Exercises
- Heegaard diagram for mnflds w/ ∂.

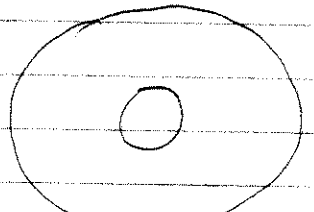
Def: A sutured Heegaard diagram is $(\Sigma, \vec{\alpha}, \vec{\beta})$

- Σ compact surface with $\partial\Sigma \neq \emptyset$ (Σ may be disconnected, but each component Σ_i satisfying $\partial \Sigma_i \neq \emptyset$)
- $\vec{\alpha} = \{\alpha_1, \dots, \alpha_k\}$ is a collection of k curves, lin. ind. in $H_1(\Sigma; \mathbb{R})$
- $\vec{\beta} = \{\beta_1, \dots, \beta_k\}$

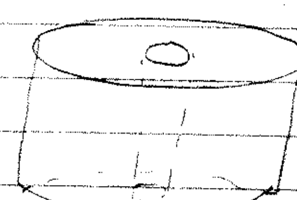
Lemma. A sutured H.D. gives rise to a 3-mnfld w/ boundary, unique up to homeomorphism.

Pf. $$\Sigma \times I \ \bigcup_{i=1}^{k} \underbrace{D^2_{\alpha_i} \times I}_{\substack{\text{Attached to} \\ \Sigma \times \{-1\}}} \ \bigcup_{i=1}^{k} \underbrace{D^2_{\beta_i} \times I}_{\substack{\text{Attached to} \\ \Sigma \times \{1\}}}$$

Ex:

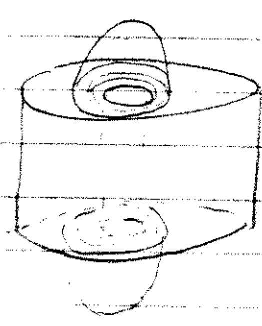


$\Sigma = S^1 \times I$

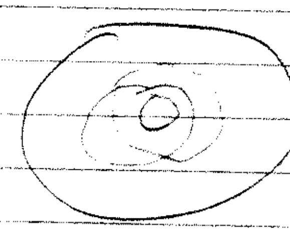


Note: $(\Sigma, \emptyset, \emptyset) \rightsquigarrow \Sigma \times I$.

Ex.



$\simeq S^2 \times I$



<u>Note</u>: The boundary of the manifolds constructed from a sutured H.D. are naturally divided into 3 pieces.

(1) $\partial\Sigma \times I$ $\quad = $ sutures

(2) $\Sigma \times \{-1\}$, surgered along $\vec{\alpha}$ $\quad =: R_-$

(3) $\Sigma \times \{1\}$, surgered along $\vec{\beta}$ $\quad =: R_+$

<u>Notice</u>: Sutures are just annuli. Often, we will conflate an annulus with its core.

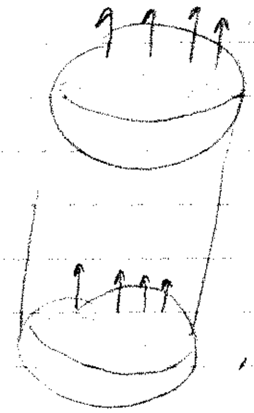
<u>Def.</u> We call a 3-mnfld. w/ boundary, partitioned in this way, a <u>sutured mnfld.</u> We'll denote it (M, γ), where γ is the sutures.

$$\partial M = R_+ \cup_\gamma R_-.$$

<u>Observe</u>: We orient ∂M in a non-standard way.

Typically, we'd orient ∂M by outward normal first convention.

Here, we're orienting ∂M s.t. R_+ and R_- inherit the orientation from $\Sigma \times I$.

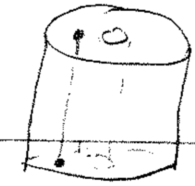


i.e. orientation along R_- (R_+) is such that oriented normal points into M along R_- (out of) (R_+).

Gabai defined sutured mnflds. more generally (so ∂M might contain some tori w/ no sutures.)

- His motivation was to construct foliations on such mnflds. so as to detect genera of homology classes.
- His technique was to cut (M, γ) along a properly embedded surface $(S, \partial S) \subset (M, \partial M)$, + construct new sutured decomposition of $(M, \partial M) \setminus (S, \partial)$
$$\shortparallel$$
$$(M', \gamma'),$$
then iterate the construction until obtaining a <u>product sutured mnfld.</u>, i.e.

<u>Def.</u> A <u>product sutured mnfld.</u> is $(\Sigma \times I, \partial\Sigma)$
$$\shortparallel$$
$$(M, \gamma)$$



Note: A product sutured mnfld. has a taut foliation,
i.e. a foliation w/ a properly embedded arc that transversely intersects every leaf.

Gabai is able to glue back product sutured mnfld. (w/ taut foliations) together
while simultaneously gluing the leaves of the foliation.

If you have a taut foliation on a mnfld. with $(S, \partial S) \hookrightarrow (M, \partial M)$
a compact leaf, then

$$-\chi(S) = \max_{S'} \left\{ -\chi(S') \;\middle|\; \substack{S' \hookrightarrow M \\ [S', \partial S'] = [S, \partial S]} \right\}$$

Lemma. Given $(\Sigma, \vec{\alpha}, \vec{\beta})$ a sutured H.D., and the associated sutured mnfld. (M, γ).
Then $\chi(R_+) = \chi(R_-)$.

Pf. $\chi(R_-) = \chi(\Sigma) + 2\underbrace{k}_{\alpha = \{\alpha_1, \ldots, \alpha_k\}} = \chi(R_+)$,

because each surgery increases χ by 2.

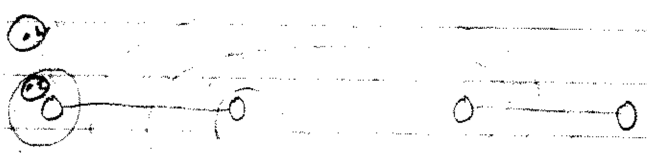
Question: Suppose $(\Sigma, \vec{\alpha}, \vec{\beta})$ is a sutured diagram, (but possibly different # of curves in $\vec{\alpha}, \vec{\beta}$).

$$\begin{array}{ccc} \mathrm{Sym}^k & (\Sigma) & \forall\, k \le \min\{|\vec{\alpha}|, |\vec{\beta}|\} \\ \cup\!| & \cup\!| & \\ \mathbb{T}_A & \mathbb{T}_B & \end{array}$$

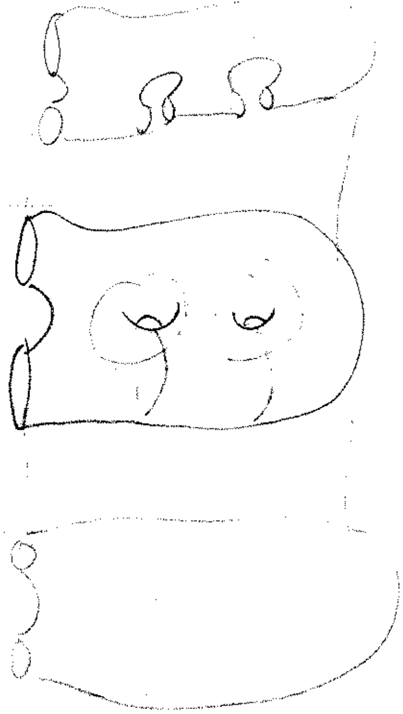
$A \subseteq \{\alpha_1, \ldots, \alpha_n\}$ $\quad |A| = |B| = k$
$B \subseteq \{\beta_1, \ldots, \beta_m\}$

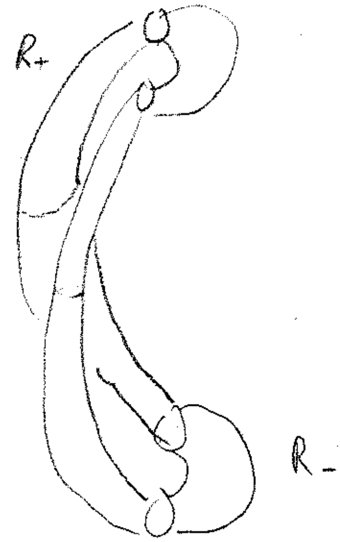
What, if anything, is $HF_*(\mathrm{Sym}^k(\Sigma), \mathbb{T}_A \cap \mathbb{T}_B)$ telling us?
e.g. For knot complements or Seifert surface complements,
do these groups contain information about the Alexander module
outside of its order (the Alexander polynomial)?

Ex:



2 pted. HD specifies a sutured H.D.
by removing a nbhd. of z and w.





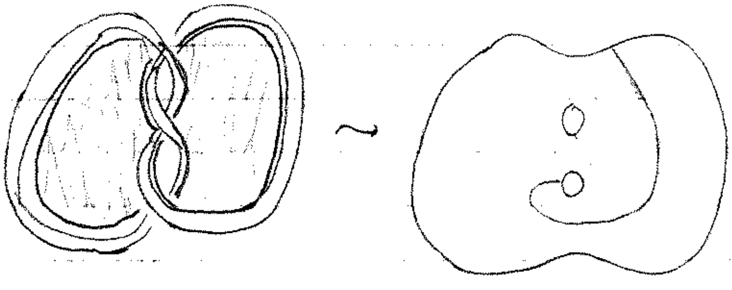
So, doubly pted H.D. gives a sutured mnfld. w/ Torus boundary + 2 parallel sutures. Call it $(Y(K), \gamma_2)$

__Exercise__: Check that $(Y(K), \gamma_2) \cong (Y - \text{nbhd}(K), \text{ 2 meridional sutures})$

__Ex__: Let $F \subseteq Y - \text{nbhd}(K)$ be a Seifert surface.

$(Y(K) - \text{nbhd}(F), \partial F)$

$((S^3 - \text{Trefoil}) \setminus \text{Seifert surface})$

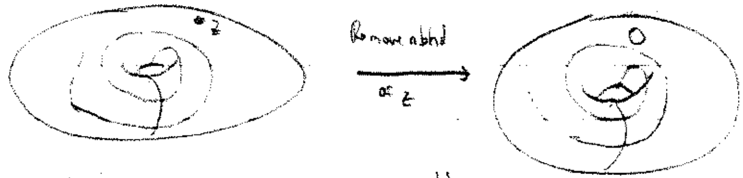


genus 2 handlebody w/ a curve (suture) on it.

__Ex__: Y closed 3mnfld.

Consider $(M = Y \setminus B^3, \gamma = \text{s.c.c. on } \partial B)$

Note: HD for $(Y \setminus B^3, \gamma)$ is simply a pted HD for Y^3 in the sense of this class.



(See Andras Juhasz' "Holo. disks + sutured mnflds.")

Given $(\Sigma, \vec{\alpha}, \vec{\beta})$ sutured H.D.,

Def. $C(\Sigma, \vec{\alpha}, \vec{\beta}) = \bigoplus_{\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \subseteq Sym^k(\Sigma)} \mathbb{F}\langle \vec{x} \rangle$

$$\partial \vec{x} = \sum_{\substack{\phi \in \pi_2(\vec{x}, \vec{y}) \\ \mu(\phi) = 1 \\ \vec{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta}} \# \widehat{\mathcal{M}}(\phi) \cdot \vec{y}$$

Thm. (Juhasz)

- $\partial^2 = 0$
- Any (balanced, i.e. $\chi(R_+) = \chi(R_-)$) sutured mfld. has a sutured H.D., and if two sutured H.D. $(\Sigma, \vec{\alpha}, \vec{\beta})$; $(\Sigma', \vec{\alpha}', \vec{\beta}')$ specify equivalent (diffeomorphic) sutured manifolds, then
$$H_*(C(\Sigma, \vec{\alpha}, \vec{\beta})) \cong H_*(C(\Sigma', \vec{\alpha}', \vec{\beta}')).$$

(class of sutured mflds. on which Reidemeister torsion is defined)

Def. $SFH(M, \gamma) := H_*(C(\Sigma, \vec{\alpha}, \vec{\beta}), \partial)$ for some $(\Sigma, \vec{\alpha}, \vec{\beta})$ specifying (M, γ).

Pf. $\partial^2 = 0$ is easy. (Just like before, but easier because we're counting fewer hyper-surfaces.)

Uses Gromov compactness lite:

In $\widehat{HF}$ or $\widehat{SFH}$, avoiding $V_z = z \times Sym^{k-1}(\Sigma)$ for any 1 basepoint

$\Rightarrow$ No J-holomorphic sphere bubbling.

Badness happens when $\pi_2(Sym^k(\Sigma)) \neq 0$

$\pi_2(Sym^k(\Sigma), \mathbb{T}_\alpha) \neq 0$

$\pi_2(Sym^k(\Sigma), \mathbb{T}_\beta) \neq 0$.

Here, $\pi_2(Sym^k(\Sigma)) \cong \mathbb{Z}\langle S \rangle$ s.t.

$S \cap V_z = \pm 1 \quad \forall z \in \Sigma$.

$\pi_2(Sym^k(\Sigma), \mathbb{T}_\alpha) \cong \mathbb{Z}\langle S' \rangle$

$ \mathbb{T}_\beta) \cong \mathbb{Z}\langle S' \rangle$.

Invariance Pf.

We already proved this.

Sketch: Any two diagrams for (M, γ) can be connected by
- isotopies
- handleslides
- stabilizations

$\Big\}$ Cerf theory

Verified that these moves don't change $SFH(Y^3 - B, \gamma)$

$$SFH(Y^3 - B, \gamma) = \widehat{HF}(Y)$$

Matt Hedden HFH 4/12/11

Last Time:

- Defined $SFH = H_*\left(C(\Sigma,\vec{\alpha},\vec{\beta}),\ \partial\right)$, where

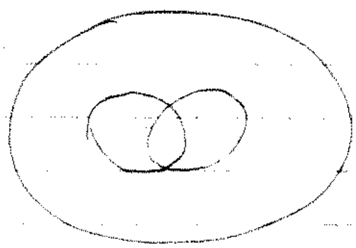
$(\Sigma,\vec{\alpha},\vec{\beta})$ is a Heegaard diagram adapted to a balanced sutured mnfld. (i.e. $\chi(R_+) = \chi(R_-)$)

($C(\Sigma,\vec{\alpha},\vec{\beta})$ comes from $\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \subseteq Sym^{|\vec{\alpha}|=|\vec{\beta}|}\Sigma$.)

Note, again, our sutured mnfld. satisfy $\pi_0(\gamma) \twoheadrightarrow \pi_0(\partial M)$

Concretely, this restriction means that all components of $(\Sigma \setminus \vec{\alpha})$ & $(\Sigma \setminus \vec{\beta})$ contain $\partial\Sigma$.

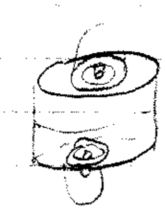
Ex:



$\Sigma = D^2$

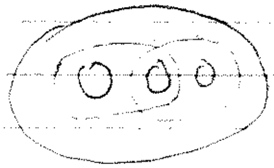
$\Sigma \setminus \vec{\alpha} =$  $\cup$ 

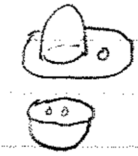
This condition is not met.

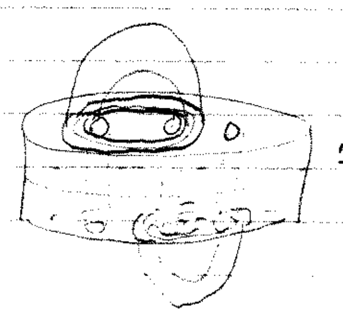
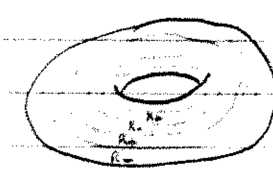
So, we get , which has sphere boundary components with no sutures. We don't allow this.

Ex:

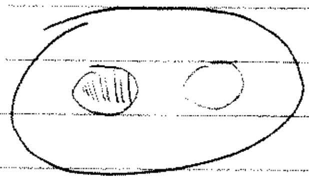
Σ = three punctured disk



R_+  $\xrightarrow{\text{surger}}$ 

$\longrightarrow$  $\simeq$ 

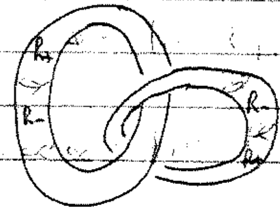
Note: $(M, \gamma) \simeq (M', \gamma')$
if $\exists$ diffeomorphism of M to M'
sending γ to γ'
orientation-preserving



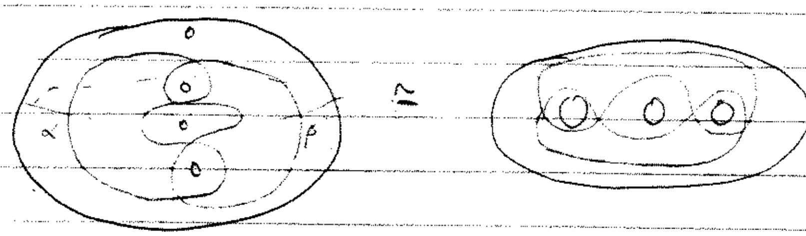
$\rightsquigarrow \mathrm{Sym}^1(D^2)$
$\cup\!\!\!\mid \quad \cup\!\!\!\mid$
$\alpha \quad \beta$

$\pi_2(D) \cong 0$
But $\pi_2(D, \alpha) \cong \pi_2(D, \beta) \cong \mathbb{Z}$
Having $\pi_2(M^{2n}$, Lagrangian$) \neq 0$ is bad for Floer homology
due to "bubbling"
This is why it's important that every boundary component contains sutures.

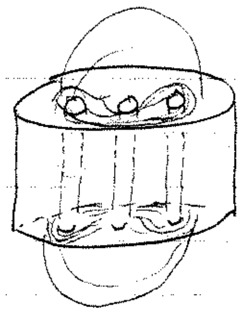
Observation: A link complement can be equipped w/ a sutured mnfld. structure by taking 2 parallel meridional sutures on each boundary.



Drawing a H.D. for a link complement in this way,



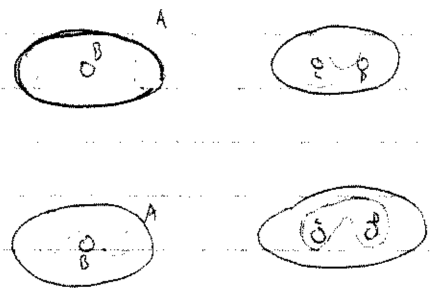
$\rightsquigarrow$



One can see the Hopf link by tracking a longitude for each torus boundary component intersecting each suture once.

Note that they link once.

∂:



Define: $\widehat{HFL}(L) := SFH(Y - n(L), \gamma)$

two parallel meridional sutures on each component.

Thm. $\chi(\widehat{HFL}(L))$

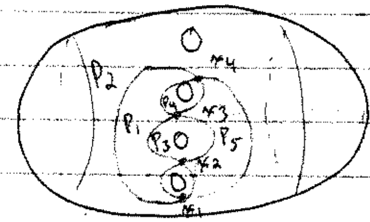
$$= \prod_{i=1}^{|L|} \left(T_i^{\frac{1}{2}} - T_i^{-\frac{1}{2}}\right) \cdot \Delta_L(T_1, \ldots, T_{|L|})$$

For $|L|>1$

$$\pi_1(S^3 - n(L)) \xrightarrow{\phi} H_1(S^3 - L) \cong \mathbb{Z}^{|L|}$$ $\leftarrow$ generated by meridians

$$\Delta_L = \det\left(\text{presentation matrix for } H_1\left(X^L_\infty ; \mathbb{Z}[T_1, T_1^{-1}, \ldots, T_{|L|}, T_{|L|}^{-1}]\right)\right)$$

Ex: (Hopf Link, cont'd.)



$$\pi_2(x_i, x_j) \cong \begin{cases} \emptyset & \varepsilon(x_i, x_j) \neq 0 \\ ? & \varepsilon(x_i, x_j) = 0 \end{cases}$$

If $\underset{\psi}{\pi_2(x_i, x_j)} \neq \emptyset \Rightarrow$

$$\left\{ \begin{tikzcd} F^2 \arrow[r, "\Phi"] \arrow[d, "\pi", "g=1"'] & \Sigma \\ \mathbb{D}^2 & \end{tikzcd} \right\} \rightsquigarrow D(\psi) = \sum n_{p_i}(\psi) \cdot D_i$$

$$\partial D(\psi) = \gamma_\alpha \cup \gamma_\beta =: \gamma$$

collection of arcs connecting $\vec{x}$ to $\vec{y}$ along α

———— $\vec{y}$ to $\vec{x}$ — β

$\gamma = \partial D(\psi)$, i.e. is null-homologous.

In the case at hand, $D(\psi)$ has $n_{p_i}(D(\psi)) = 0$ in all P_i regions adjacent to $\partial\Sigma$.
(Think of $\partial\Sigma$ as basepoints.)

$$\varepsilon(\vec{x}, \vec{y}) = [\gamma] \in H_1(\Sigma) / \text{Span}\,\alpha + \text{Span}\,\beta$$

Can this be the boundary of a domain of a Whitney disk?

$$\cancel{H_1(\Sigma, \partial\Sigma) / \text{Span}\,\alpha + \text{Span}\,\beta}$$

So $\varepsilon(x_i, x_j) \in H_1(M)$.

What if $\varepsilon(\vec{x}, \vec{y}) = 0 \in H_1(M)$?

Then we can find $D(\psi)$ s.t. $\partial D(\psi) = \gamma + \sum n_i \alpha_i + \sum m_i \beta_i$
+ $D(\psi)$ is domain of Whitney disk.

How many Whitney disks are there?

Suppose $\psi, \psi' \in \pi_2(\vec{x}, \vec{y})$.

Then $\psi' * \psi^{-1} \in \pi_2(\vec{y}, \vec{y}) = \{$ null-homologies between $\vec{\alpha} + \vec{\beta}$ curves in $H_1(\Sigma)$ $\}$

i.e. $\mathrm{Ker}(\mathrm{Span}\,\vec{\alpha} + \mathrm{Span}\,\vec{\beta} \to H_1(\Sigma))$

i.e. Periodic domains.

$(M = Y)$ Mayer-Vietoris $\nearrow$ $\cong H_2(M)$.

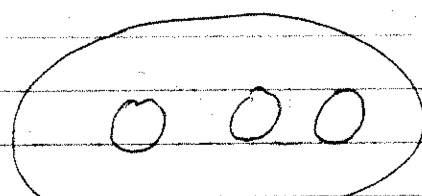
Lemma. $\pi_2(\vec{x}, \vec{y}) = \begin{cases} \emptyset & \varepsilon(\vec{x}, \vec{y}) \neq 0 \in H_1(M) \\ H_2(M) & \varepsilon(\vec{x}, \vec{y}) = 0 \end{cases}$

$\uparrow$ No Extra $\mathbb{Z}$, because the characteristic class of the surface is in $H_2(\Sigma, \partial\Sigma)$, not $H_2(\Sigma)$

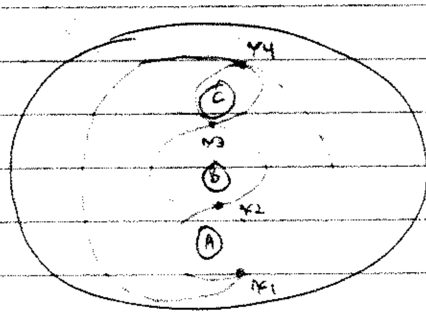
$$H_2(\Sigma, \partial\Sigma) \cong \mathbb{Z}\langle[\Sigma, \partial\Sigma]\rangle.$$

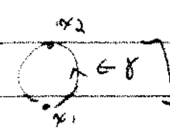
$n_{p_i}([\Sigma, \partial\Sigma]) \neq 0$ for p_i adjacent to $\partial\Sigma$.

$$H_1(M) \cong H_1(\Sigma)/_{\mathrm{Span}\,\alpha + \mathrm{Span}\,\beta} \cong \mathbb{Z}\langle A\rangle \oplus \mathbb{Z}\langle B\rangle \oplus \mathbb{Z}\langle C\rangle \Big/ \begin{matrix}\langle A+C=0\rangle \\ \langle -A-C=0\rangle\end{matrix}$$

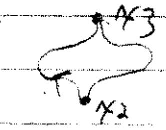


$$\cong \mathbb{Z}\langle A = -C\rangle \oplus \mathbb{Z}\langle B\rangle$$



$\varepsilon(x_1, x_2) = [$  $\in \gamma] \in H_1(M)$

$= [A] \neq 0$.

$\varepsilon(x_2, x_3) = \Bigg[$  $\Bigg] = [B] \neq 0.$

$\varepsilon(x_3, x_4) = \Bigg[$  $\Bigg] = [C] \neq 0$

~~Since Whitney disks are transitive~~

If $\varepsilon(\vec{x}, \vec{y}) \neq 0$ and $\varepsilon(\vec{y}, \vec{r}) \neq 0 \not\Rightarrow \varepsilon(\vec{x}, \vec{r}) \neq 0$.

Ex: $\varepsilon(\vec{x}, \vec{y}) \neq 0$, $\varepsilon(\vec{y}, \vec{x}) \neq 0$, but $\varepsilon(\vec{x}, \vec{x}) = 0$.

But ε-class is additive, so we can conclude that there are no Whitney disks at all!

$$SFH(S^3 \setminus n(\text{Hopf Link}), \gamma) \cong \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}.$$

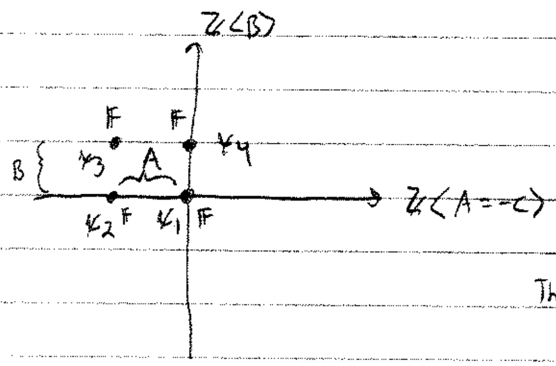
Observe: $SFH(M, \gamma) \cong \bigoplus_{\varepsilon\text{-classes}} SFH(M, \gamma; \varepsilon)$

ε-classes $\updownarrow$ 1-1

$H_1(M) \cong H^2(M, \partial M)$

$\cong$

$\mathbb{Z}\langle A = -C\rangle \oplus \mathbb{Z}\langle B\rangle$



The choice of placing x_1 was arbitrary, but the differences determined the relative placements of the rest of the generators.

Thus, this picture is unique up to overall translation.

Regard $\chi(SFH(M, \gamma)) \in \mathbb{Z}[H_1(M)]$. Well-defined up to mult. by units, i.e. $\pm T_A^n T_B^m$.

In this example, $\mathbb{Z}[H_1(M)] \cong \mathbb{Z}[\mathbb{Z}\langle A\rangle \oplus \mathbb{Z}\langle B\rangle]$

$\mathbb{Z}[T_A, T_A^{-1}, T_B, T_B^{-1}]$

$\chi(SFH(\text{Hopf})) = \pm T_B^1 T_A^0 \pm T_B^0 T_A^0 \pm T_B T_A^{-1} \pm T_B^0 T_A^{-1}$

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Last time: Defined $\widehat{HFL}(L)$ as $SFH(Y - n(L),$ 2 meridional sutures on each component$)$

Computed $\widehat{HFL}$(Hopf Link)

Discussed $\varepsilon(\vec{x},\vec{y}) = [\underset{\gamma_a \cup \gamma_b}{\gamma}] \in H_1(M) \cong H^2(M,\partial)$

Lemma $\varepsilon(\vec{x},\vec{y}) = 0 \Leftrightarrow \pi_2(\vec{x},\vec{y}) \neq \emptyset$

Lemma. $\varepsilon(\vec{x},\vec{y}) = 0 \Rightarrow \pi_2(\vec{x},\vec{y}) \cong H_2(M)$

Note: For $\widehat{HFL}(L \subseteq S^3)$, $H_2(S^3 - L) = 0$

This fact gives a (non-canonical) splitting

$$SFH(M,\gamma) \cong \bigoplus_{\alpha \in H_1(M)} SFH(M,\gamma;\alpha)$$

Given $Y^3 \xrightarrow{\text{remove } B^3} (Y \setminus B^3, \gamma)$

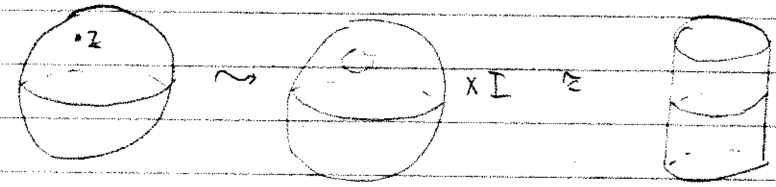
$SFH(Y \setminus B^3, \gamma) \cong \widehat{HF}(Y)$

and $H_1(Y \setminus B^3) \cong H_1(Y)$

(No new information.
One has basepoint, one has boundary component
Otherwise, same H.D.)

$Y^3 \xrightarrow[B^3\text{'s}]{\text{remove two}} (Y^3(2), \gamma)$

HD for $(Y(2), \gamma)$



$$pt. = Sym^0(\Sigma) \supseteq Sym^0(\phi_\alpha), \quad \supseteq Sym^0(\phi_\beta)$$

$$SFH(S^3(1), \gamma) \cong \mathbb{Z}\langle pt. \rangle$$

Let (M,γ) be $(\Sigma \times I, \partial\Sigma \times \{0\})$ a sutured manifold. $\Rightarrow SFH(M,\gamma) \cong \mathbb{Z}$.

This fact will, ultimately, prove that $\widehat{HFK}(K, g(K)) \cong \mathbb{Z}$ for a fibered knot.

Exercise: Show this. (Recall: $HF^+(Y|_F, 2g-2) \cong \mathbb{Z}$ for $F^2 \hookrightarrow Y \to S^1$)

Hint: Construct H.D. from the fibration $F^2 \times I \cup F^2 \times I$

Thm. $SFH(Y \setminus K, \underset{\text{meridians}}{2 \mu}) \cong \widehat{HFK}(Y, K)$

Pf. We did it!

$$\begin{aligned}\widehat{CFK}(Y,K) &= \left(C(\Sigma, \vec{\alpha}, \vec{\beta}, z, w), \hat{\partial}\right) \\ &= \left(C(\Sigma \setminus n(z,w), \vec{\alpha}, \vec{\beta}), \partial^{SFH}\right) \\ &= SFH(Y \setminus K, \mu)\end{aligned}$$

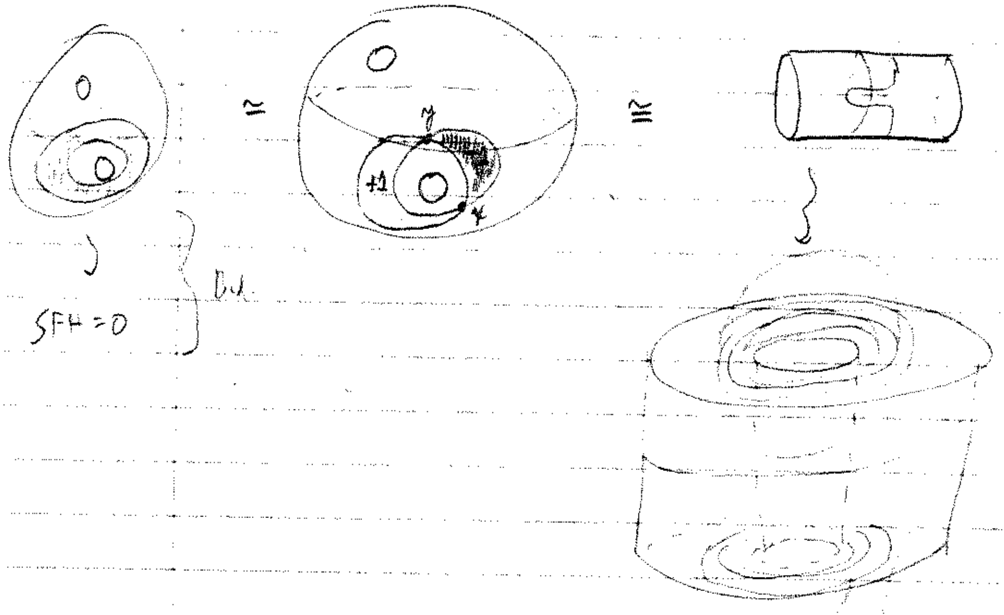
$$(Y \setminus n(K), \mu) \xrightarrow[\substack{\text{along} \\ \text{Seifert surface}}]{\text{decompose}} \left((Y \setminus n(K)) \setminus F, \partial F\right)$$

$$\left((Y \setminus n(K)) \setminus F\right) \cong Y \setminus n(F)$$

Thm. $SFH(Y \setminus n(F), \partial F) \cong SFH(Y \setminus n(K), \mu; g(F))$

$g(F)$: Spin^c- or Alexander grading

HD for $(S^3(2), \gamma_2)$?



Did.

$SFH = 0$

$\cong (S^3(2), \gamma_2)$.

$\cong S^2 \times I$

$C(\Sigma, \alpha, \beta) \cong \mathbb{F}\langle x \rangle \oplus \mathbb{F}\langle y \rangle$

$\varepsilon(x, y) = 0$

$H_2(S^3(2), \gamma_2) \cong \mathbb{Z}$

$\exists$ 2 Whitney disks from y to x, each with unique J-holomorphic reps. (by RMT)

$\partial y = 2x = 0 \in \mathbb{F}$.

$gr(\vec{y}) - gr(\vec{x}) = \mu(\phi)$

$\Rightarrow SFH(S^3(2), \gamma_2) \cong V = \mathbb{F}_i \oplus \mathbb{F}_{i-1}$

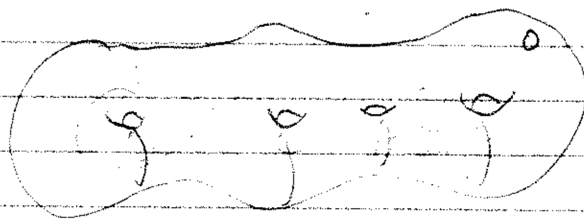
nearly the same V as in Ciprian's talk

Note: If $H_2(M) \neq 0$, we need an admissibility hypothesis

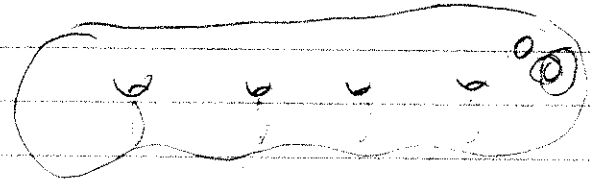
We need weak admissibility, i.e.

Every periodic domain has both positive and negative coefficients.

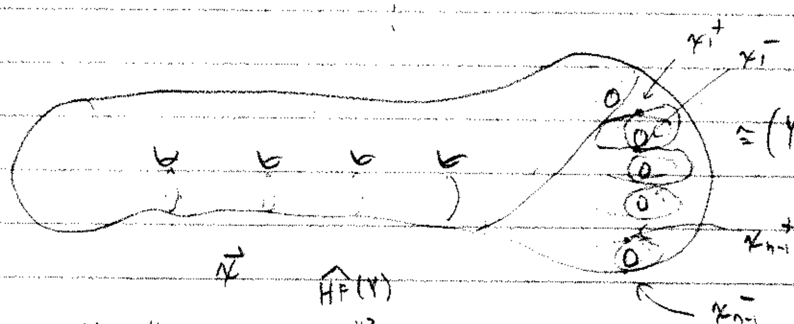
Ex: $(Y^3(n), \gamma_n)$



$\cong (Y(1), \gamma_1)$



$\cong (Y(2), \gamma_2)$



$\cong (Y(n), \gamma_n)$

$\vec{x}$ $\qquad \widehat{HF}(Y)$

$\xrightarrow[\text{Formula}]{\text{Kunneth}} \quad SFH(Y(n), \gamma_n) \cong SFH(Y(1), \gamma_1) \otimes V^{n-1}$

$\vec{x} \otimes x_1^{\pm} \otimes \cdots \otimes x_{n-1}^{\pm}$

Def. An n-pted H.D. for Y is $(\Sigma_g, \{\alpha_1, \dots, \alpha_{g+n-1}\}, \{\beta_1, \dots, \beta_{g+n-1}\}, w_1, \dots, w_n)$

s.t. (1) Σ closed, oriented surface.

(2) $\vec{\alpha}$ disjoint, s.c.c. spanning $\mathbb{R}^g \subseteq H_1(\Sigma_g, \mathbb{R}) \cong \mathbb{R}^{2g}$

(3) $\vec{\beta}$ ______

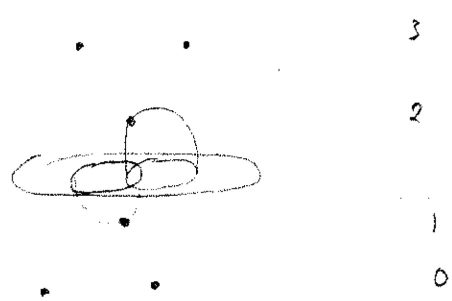
(4) Every component of $\Sigma \setminus \vec{\alpha}$ contains a w_i, and ______ $\Sigma \setminus \vec{\beta}$ ______.

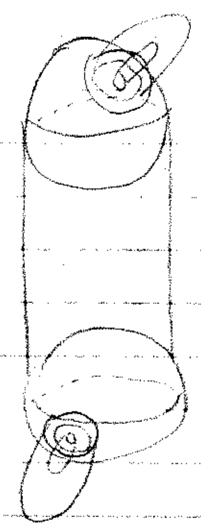
Observe: These HD's can be interpreted / constructed by Morse theory through self-indexing Morse functions s.t. $\exists$ $g+n-1$ index 1 critical pts.

$g+n-1$ — 2 —

n — 0 —

n — 3 —





$\cup\ 2B^3$'s

From such an n-pointed HD, we can define Floer complexes

$$\widetilde{C}(\Sigma,\vec{\alpha},\vec{\beta},\vec{w}) = \bigoplus_{\vec{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \mathbb{F}\langle\vec{x}\rangle$$

$$\widetilde{\partial}\vec{x} = \sum_{\vec{y}} \sum_{\substack{\phi \\ n_{w_i}(\phi)=0\ \forall i=1,\dots,n}} \#\mathcal{M}(\phi)\cdot\vec{y}.$$

$SFH(Y(n),\gamma_n)$

$\cong$

$H_*(\widetilde{C},\widetilde{\partial})$ is not an invariant of Y - it is isomorphic to $\widehat{HF}(Y)\otimes V^{n-1}$

which IS an invariant of $(Y(n),\gamma_n)$

$$C^-(\Sigma, \vec{\alpha}, \vec{\beta}, \vec{w}) = \bigoplus_{\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \mathbb{F}[U_1, \ldots, U_n]\langle \vec{x} \rangle \; ; \quad \partial^-(x) = \sum_{y} \sum_{\phi} \# M(\phi) \cdot U_1^{n_{w_1}(\phi)} \cdots U_n^{n_{w_n}(\phi)} \cdot \vec{y}$$

etc.

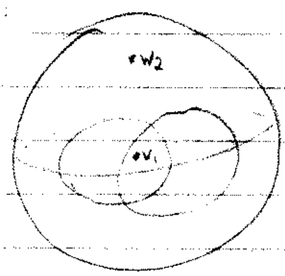
__Thm.__ $\partial^- \circ \partial^- = 0$.

and $H_*(C^-, \partial^-) \cong HF^-(Y)$

$$\begin{array}{ccc} H_*(C^-, \partial^-) & \cong & HF^-(Y) \\ \circlearrowleft & & \circlearrowleft \\ \mathbb{F}[U_1, \ldots, U_n] & & \mathbb{F}[U] \\ \circlearrowleft & & \\ \mathbb{F}[U], \text{ where} & & (U_i)_* \cong (U_j)_* \;\; \forall i, j \\ U \text{ acts as } U_1. & & \end{array}$$

$$\begin{array}{ccc} H_*(C^-, \partial^-) & \xrightarrow{\cong} & H_*(C^-, \partial^-) \\ \big\downarrow (U_j)_* & & \big\downarrow (U_1)_* \\ H_*(C^-, \partial^-) & \xrightarrow{\cong} & H_{*-2}(C^-, \partial^-) \end{array}$$

__Exercise:__



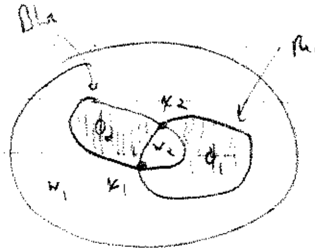
Compute (C^-, ∂^-).

Matt Hedden HFH 4/19/11

LATE

$\pi_2(x,x) \cong \mathbb{Z}\langle\Sigma\rangle \oplus H_2(Y)$

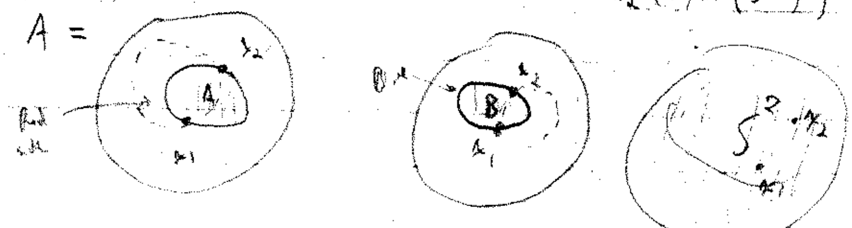
$$C^- = \mathbb{F}[U_1,U_2]\langle x_1\rangle \oplus \mathbb{F}[U_1,U_2]\langle x_2\rangle$$
$$\partial^- x_1 = 1\cdot x_2 + 1\cdot x_2 = 0$$



$$\pi_2(x,x) \cong \underset{\substack{\text{III}\\ \mathbb{Z}\ (\text{if } g>2)}}{\pi_2(Sym^g(\Sigma))} \oplus \underset{\substack{\cup\!\mid\\ H_2(Y)}}{\text{Periodic Domains}} := \ker\{Span\,\vec{\alpha} + Span\,\vec{\beta} \to H_1(\Sigma)\}$$

In Ex, $\{\text{Periodic Domains}\} \cong \mathbb{Z}\langle A\rangle \oplus \mathbb{Z}\langle B\rangle \oplus \mathbb{Z}\langle S^2\rangle$

$\leftarrow \pi_2(Sym^1(S^2))$

$A =$ 

So any $\phi \in \pi_2(x,x_2)$ differs from ϕ_1 by adding
$$nA + mB + rS^2 \quad , n,m,r \in \mathbb{Z}$$

Note: $\mu(nA+mB+rS^2) = n\mu(A) + m\mu(B) + n\mu(S^2)$

$\}$

Suppose $\phi' = \phi_1 * (nA + mB + rS)$ & $\mu(\phi') = 1$
$\Rightarrow 2n + 2m + 4r = 0.$

We also need $n_{p_i}(\phi') \geq 0 \;\; \forall p_i \in S^2 \setminus (\alpha\cup\beta)$

$$n_{p_1}(\phi') = 0 + 0 + 0 + \underset{\substack{\| \\ r}}{r\,n_{p_1}(S)}$$

So $r \geq 0.$

$n_{p_2} = 0 + m + 0 + r \geq 0$

$n_{p_3} = 0 + n + m + r \geq 0$

$n_{p_4} = 1 + 0 + n + r \geq 0$

<u>Lemma.</u> If $\mu(\phi') = 1$ & $n_{p_i} \geq 0 \ \forall i = 1, \dots, 4$,

Then $\phi' = \phi_2$ or $\phi' = \phi_1$.

<u>Pf.</u> $r \geq 0$ + $r \geq -n - m$

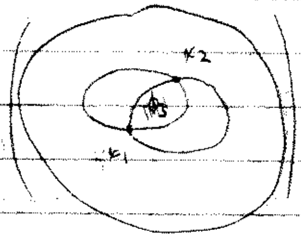
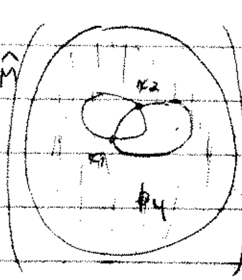
$4r + 2(n+m) = 0 \Rightarrow 2r = -n - m \Rightarrow r \geq 2r \Rightarrow r \leq 0.$

$\Rightarrow r = 0.$

Maslov index forces $n = -m$.

This now verifies that $\partial^- x_1 = 1 \cdot x_2 + 1 \cdot x_2 = 0.$

$\partial^- x_2 = ?$

<u>Lemma.</u> $\partial^- x_2 = \hat{M}\Big($  $\Big) \cdot x_1 + \hat{M}\Big($  $\Big) \cdot$

$\partial^- x_2 = \hat{M}\Big(\quad\Big) \cdot U_1^{n_{w_1}(\phi_3)} \cdot U_2^{n_{w_2}(\phi_3)} \cdot x_1 + 1 \cdot U_1^{n_{w_1}(\phi_4)} \cdot U_2^{n_{w_2}(\phi_4)} \cdot x_1$

$= U_1^0 \cdot U_2^1 \cdot x_1 + U_1 x_1$

$\partial^- x_2 = (U_2 + U_1) x_1$

<u>Pf</u> Symmetry of the previous argument which ~~[illegible]~~ ruled out all other holomorphic disks by (Maslov index + positivity)

the homology

$$\mathbb{F}[U_1, U_2]\langle x_2\rangle \xrightarrow[\;]{\overset{\partial^-}{\underset{U_1+U_2}{\shortparallel}}} \mathbb{F}[U_1, U_2]\langle x_1\rangle \xrightarrow{\partial^-} 0$$

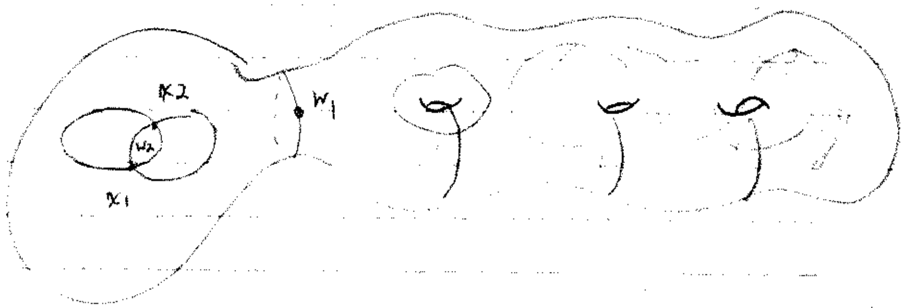
$$H_*(C^-) \cong \ker \partial^- / \operatorname{Im} \partial^-$$

$$\cong \frac{\mathbb{F}[U_1, U_2]\langle x_1\rangle}{\langle U_1 + U_2\rangle}$$

(mod 2)

$$\cong \mathbb{F}[U] \quad \cong HF^-(S^3).$$

This example is a model for the general case:



We've shown that 0-3 stabilizing the 1-pted. genus zero diagram of S^3 leaves $H_*(C^-)$ unchanged as an $\mathbb{F}[U]$-module.

Algebraically, the proof of 0-3 stab. invariance is following.

C^- original H.D.'s complex

C^-_{03} stabilized H.D's complex $\cong C^-[U_2]\langle x_1\rangle \oplus C^-[U_2]\langle x_2\rangle$

Claim: $(C^-_{03}, \partial^-) \cong \left(C^-[U_1]\langle x_1\rangle \oplus C^-[U_2], \sim \right)$

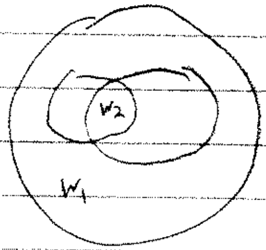
Note: $H_*(RHS) \cong H_*(C^-[U_2])$, where $(U_1)=(U_2)$.

Claim: $(C^-_{03}, \partial^-) \cong$

$M(U_1 + U_2)$ ← mapping cone complex of $U_1 + U_2 : (C^-[U_2], \partial^-) \to (C^-[U_2], \partial^-)$

and $H_*(M(U_1+U_2)) \cong H_*(C^-) \circlearrowleft \mathbb{F}[U]$.

$\circlearrowleft \mathbb{F}[U_2]$



$\tilde{\partial} \equiv 0$

↑ counts only disks avoiding all basepoints.

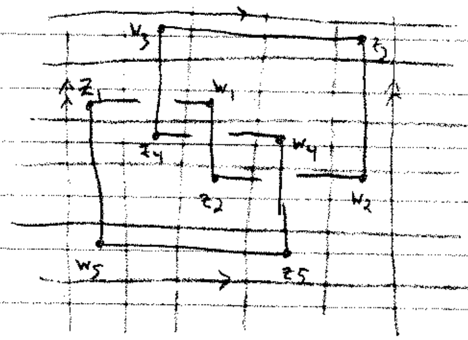
$H_*(\tilde{C}, \tilde{\partial}) \cong \mathbb{F}_{(+)} \oplus \mathbb{F}_{(-1)} \cong \widehat{HF}(S^3) \otimes V$.

Exercise: (Algebraic)

Show that Claim $\Rightarrow$ $\widehat{HF}\left((\Sigma, \vec{\alpha}, \vec{\beta}, \overset{\{w_1, \dots, w_n\}}{\overset{\shortparallel}{\vec{w}}})\right) \cong \widehat{HF}(Y) \otimes V^{n-1}$

Application

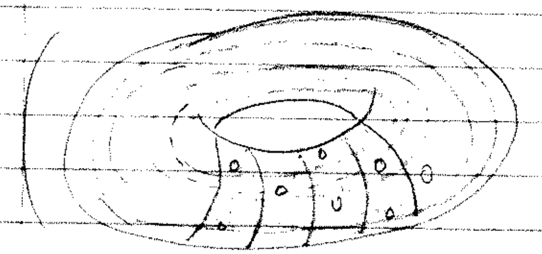
Grid diagram calculation for HFK (or HFL)



← Sutured Heegaard diagram for $(S^3 \setminus K, \gamma_{10})$

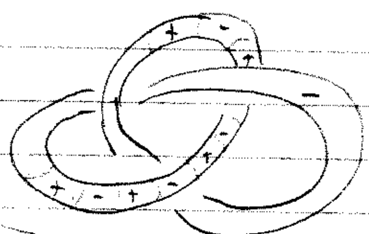
↑

5 pairs of meridional sutures.



$\times I \Big) \bigcup_{i=1}^{5} D_{\alpha_i} \bigcup_{i=1}^{5} D_{\beta_i}$

$\cong S^3 \setminus$ Trefoil



Disks miss all basepoints / sutures

Prop. $\widetilde{CFK}(\text{Diagram}, \tilde{\partial}) = SFH(S^3 \setminus K, \gamma_{10})$

$\tilde{\partial} = \partial_{SFH}$ counts empty rectangles ⟵ !!! We know the differential!

$\widetilde{CFK} = \bigoplus_{\sigma \in S_5} \mathbb{F}\langle\sigma\rangle$ ⟵ Symmetric group on $|\vec{\alpha}|$.

Pf. The only regions that can give holomorphic maps are rectangles
(Recall: Discussion of nice diagrams from last semester.)

Matt Hedden 4/21/11 HFH

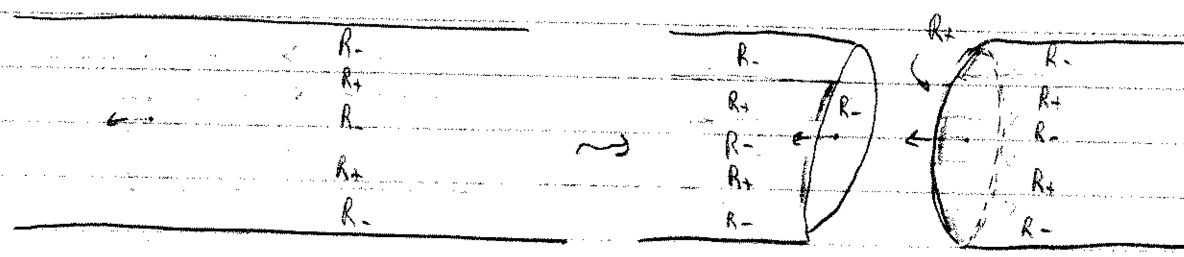
Back to SFH

Goal: Understand how SFH behaves under surface decompositions.

Roughly, A surface decomp. $(M,\gamma) \xrightarrow{S} (M',\gamma')$
cuts (M,γ) along nbhd(S) & obtains a new sutured mfld. (M',γ').

More precisely, let S be a properly embedded oriented surface in (M,γ)

$$M' \cong M \setminus \text{nbhd}(S), \qquad \partial M' = (\partial M \setminus \text{nbhd}(\partial S)) \cup \underbrace{S_+ \cup S_-}_{\partial\, \text{nbhd}(S) = \partial(S\times I)}$$



γ' = Unique extension of $\gamma \setminus \text{nbhd}(\partial S)$ to curves s.t. R_+' & R_-' don't meet.

Recall: Product sutured mflds are the simplest sutured mflds.

Def. A sutured manifold hierarchy (for (M,γ)) is a finite sequence $(M,\gamma) = (M_1,\gamma_1) \xrightarrow{S_1} (M_2,\gamma_2) \xrightarrow{S_2} (M_3,\gamma_3) \xrightarrow{S_3} \cdots \xrightarrow{S_{n-1}} (M_n,\gamma_n)$
s.t. (M_n,γ_n) is a product (i.e. $(R\times I, \partial R\times I)$).

Def. Given $(\overset{\text{connected}}{S}, \partial S) \hookrightarrow (M^3, \partial M)$, define
$$\chi_-(S) = \max\{0, -\chi(S)\}.$$
Call this the complexity of S.
For S disconnected, $\chi_-(S) = \sum_i \chi_-(S_i)$ for $S = S_1 \cup S_2 \cup \cdots \cup S_n$.

Def. Given $\alpha \in H_2(M,N)$, define $\Phi(\alpha) = \min\left\{ \chi_-(S) \;\middle|\; \begin{array}{l} (S,\partial) \hookrightarrow (M,N) \text{ satisfies} \\ i_*(S,\partial) = \alpha \end{array} \right\}$

Prop. (Thurston)

$\Phi : H_2(M,N) \longrightarrow \mathbb{Z}^{\geq 0}$ extends in a unique way to

$\Phi : H_2(M,N;\mathbb{R}) \to \mathbb{R}^{\geq 0}$, a semi-norm.

(It is a norm if we don't have any $\alpha \in H_2(M,N)$ s.t. $\alpha \neq 0$ but $\Phi(\alpha) = 0$.)

e.g. if $N = \emptyset$, + $\alpha = [T^2] \neq 0 \in H_2(T^3 = M)$, then $\Phi(\alpha) = 0$.)

Def. A balanced sutured mfld. is taut if it is irreducible (i.e. every $S^2 \hookrightarrow M$ bounds a ball) and R_+ (or R_-) are Thurston norm minimizing in $H_2(M, \partial)$.

i.e. $\Phi([R_+]) = \chi_-(R_+)$

Thm. If $\exists$ a sutured mfld. hierarchy for (M,γ), then (M,γ) is taut.

Pf. (1) A product sutured manifold is taut.

~~Immediately true for $D^2 \times I$, as $\chi_-(D^2) = 0$.~~

~~Now by induction~~

~~Cut $R \times I$ along disk arc $\times I$~~

(2) If $(M,\gamma) \overset{S}{\rightsquigarrow} (M',\gamma')$ + (M',γ') is taut, then (M,γ) is taut.

Thm. (Much harder) If (M,γ) is taut, then $\exists$ a hierarchy.

Black-boxed.

Our goal, then, is to understand how $SFH(M,\gamma)$ changes under surface decomposition.

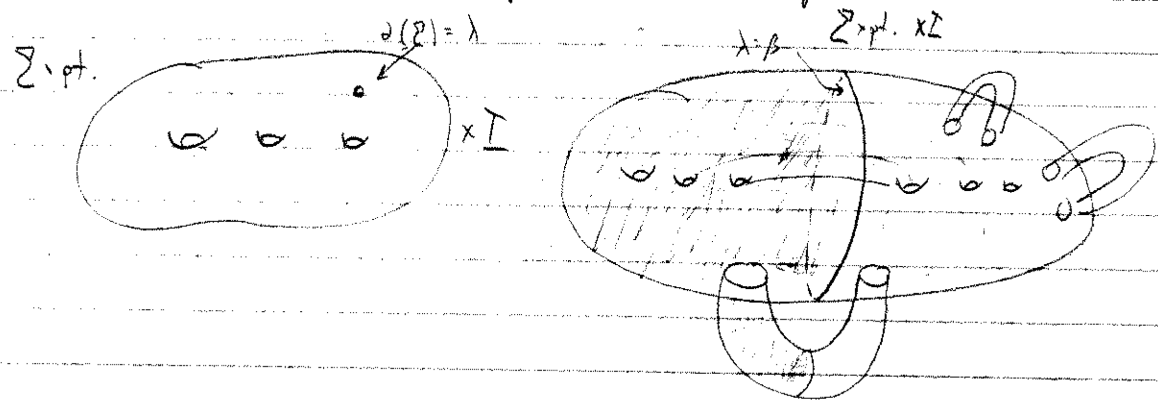
We should try to understand surface decomposition on the level of Heegaard diagrams.

"Products aren't interesting is the moral of Topology."

—Matt Hedden.

Recall: Proof of the Adjunction Inequality for HF^+ (Closed 3-mfld.)

Idea of that proof was (1) Given $\Sigma^2 \hookrightarrow Y$, we represented $\Sigma^2 - \{2 \text{ disks}\}$ as a periodic domain on a Heegaard surface representing Y.

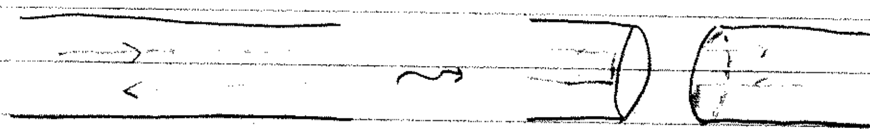


$$\langle c_1(s_w(\vec{x})), [P] \rangle = \chi(\Sigma \setminus 2\text{pts.}) + 2n_{\vec{x}}(P)$$

$[P]$ corresponds to $[\Sigma]$, and $\chi(\Sigma \setminus 2\text{pts.}) = -2g(\Sigma)$.

In SFH, we want to see the decomposing surface on the Heegaard surface.

Def. A **product disk** is a properly embedded disk in (M, γ) intersecting γ in exactly two points.

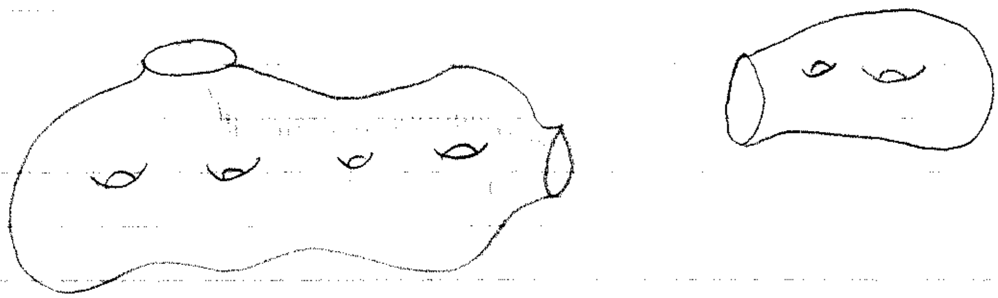


Lemma Let $(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$ be a product disk decomposition. Then (M', γ') is a product $\Leftrightarrow$ (M, γ) is a product.

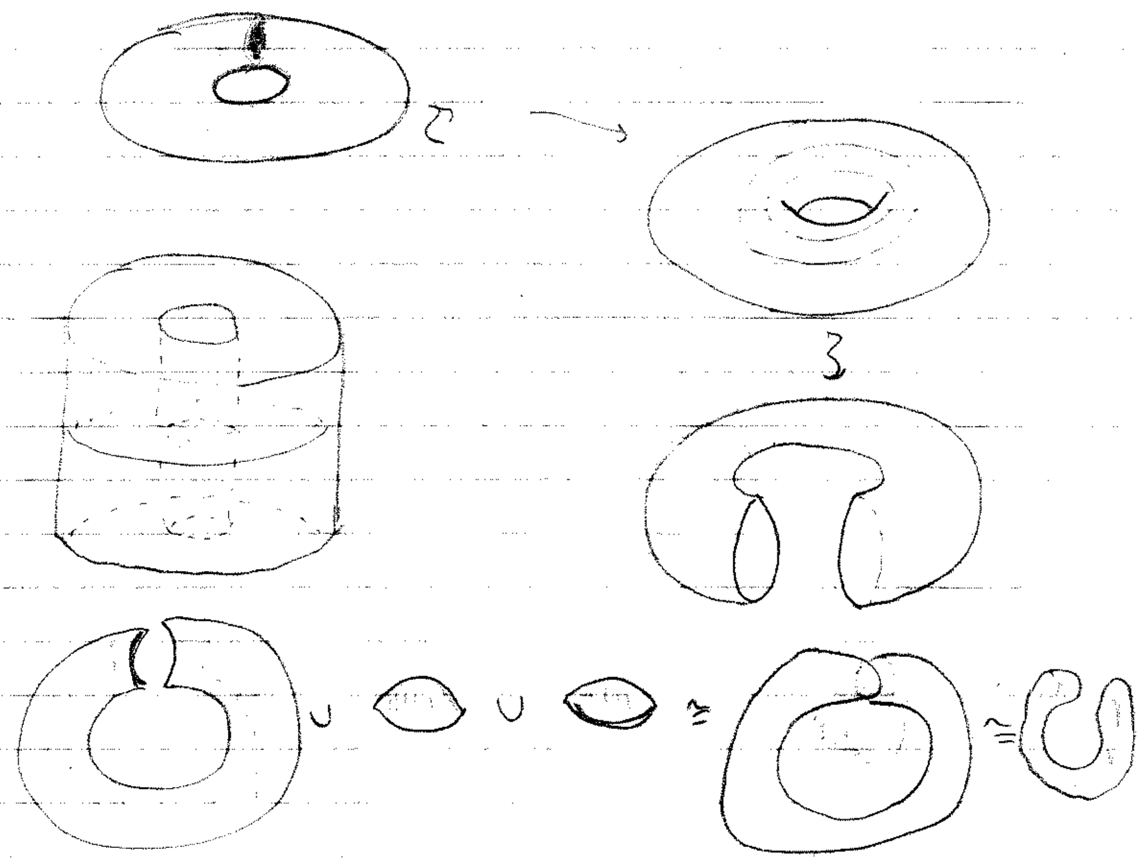
Thm. Let $(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$ be a product disk decomposition. Then $SFH(M', \gamma') \cong SFH(M, \gamma)$.

Remark: If this weren't true, we'd be screwed if we wanted to show that SFH detects products.

Pf. Given $(D^2, \partial) \hookrightarrow (M, \gamma)$, $\exists$ a H.D. for (M, γ) st.



$\exists$ bigon P with $\partial P = A \cup B$,
$A \cap \vec{\beta} = \phi$
$B \cap \vec{\alpha} = \phi$,
and $D = P \cup \left(A \times [0,1] \subseteq \Sigma \times I \cup D_\beta \right) \cup \left(B \times [-1,0] \subseteq \Sigma \times I \cup D_\alpha \right)$



Now we prove:

Step 1. Given $D \subseteq M$, find H.D. where D arises as a digon in this way.

Step 2. Show that disk decomposition corresponds to H.D. decomposition.

Step 3. $\therefore SFH(HD^{\text{before}}) = SFH(HD^{\text{after}})$

$A + B$ are isotopic, so $A + B \cap (\vec{\alpha} \cup \vec{\beta}) = \phi$.

Thus, the generators of SFH are the same before and after.

Since the differential doesn't count any disks that are adjacent to the boundary, the differential doesn't change either.

Matt Hedden HFH 4/26/11

Last time:

Def. Sutured mfld. (surface) decomp. $(M,\gamma) \xrightarrow{S} (M',\gamma')$.

Def. Thurston semi-norm on $H_2(M,\partial M)$

Def. Taut sutured mfld. ((M,γ) s.t. R_+ / R_- are Thurston norm-minimizing)

Def. Sutured mfld. hierarchy

$$(M,\gamma) \xrightarrow{S_1} \cdots \xrightarrow{S_n} (M_{n+1}, \gamma_{n+1}) \cong (R\times I,\ \partial R\times I)$$

Thm. A If $\exists$ sutured hierarchy for (M,γ), then (M,γ) is taut

Thm. B. If (M,γ) is taut, then $\exists$ a hierarchy.

Def. A product disk (PD) decomposition is

Lemma. $(M,\gamma) \xrightarrow{P.D.} (M',\gamma')$

(M',γ') taut $\iff$ (M,γ) taut

product $\iff$ product.

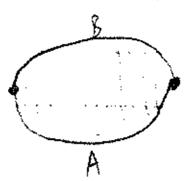
See Juhasz - Holomorphic disks & Sutured Mflds. (AGT)

- Floer Homology & Surface Decomps. (GT)

Thm. (Juhasz) $(M,\gamma) \xrightarrow{P.D.} (M',\gamma') \Rightarrow SFH(M,\gamma) = SFH(M',\gamma')$

Pf. Adapted a H.D. for (M,γ) to the product disk

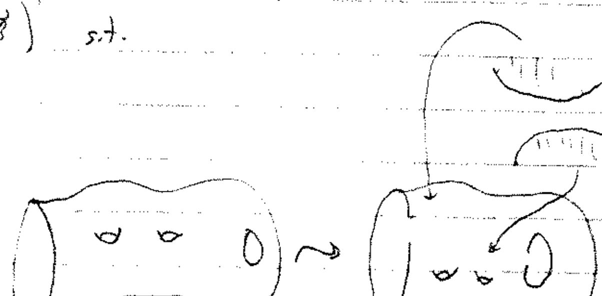
i.e. found a bigon in $(\Sigma, \vec{\alpha}, \vec{\beta})$ s.t.



(1) Vertices $\subseteq \partial\Sigma$

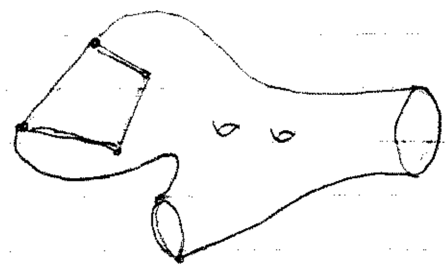
(2) $A \cap \vec{\beta} = \emptyset$

(3) $B \cap \vec{\alpha} = \emptyset$



Argued that $(M,\gamma) \xrightarrow{P.D.} (M',\gamma')$

Def. A quasi-polygon is a surface P s.t. ∂P = union of circles decomposed into edges meeting along vertices.



Lemma Suppose $(\Sigma, \vec{\alpha}, \vec{\beta})$ is a H.D. for (M, γ) + $P \subseteq \Sigma$ is a subsurface of Σ s.t.

(1) P is a quasi-polygon with each component of ∂P having an even # of vertices alternatingly labeled A + B

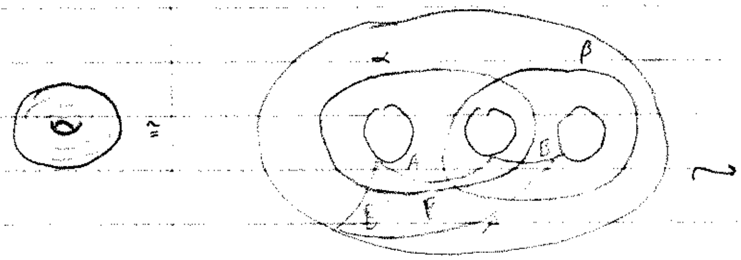
(2) Vertices of ∂P = $P \cap \partial \Sigma$.

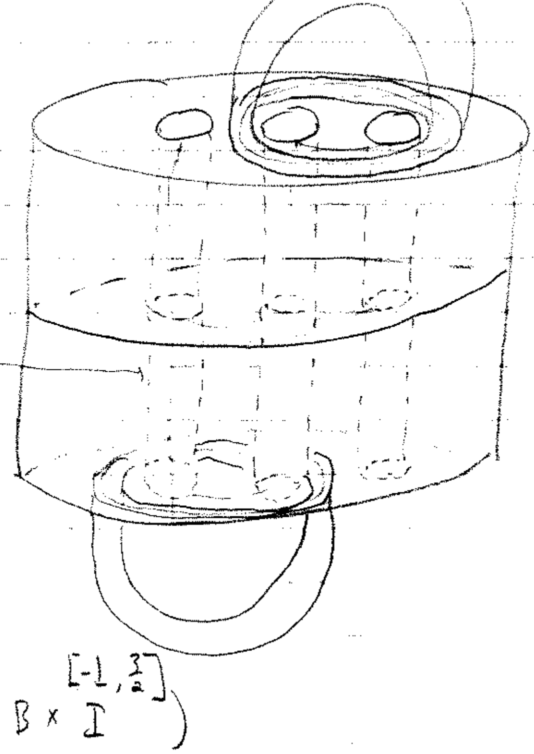
(3) $A \cap \vec{\beta} = \phi$

(4) $B \cap \vec{\alpha} = \phi$.

Then, we can construct a decomposing surface $S(P) \subseteq (M, \gamma)$ in a unique way (up to isotopy through decomposing surfaces)

Pf. (By examples, essentially)





(Saddle surface)

The A-arcs naturally flow down to the α-handle attachment sides.

B-arcs ——— up

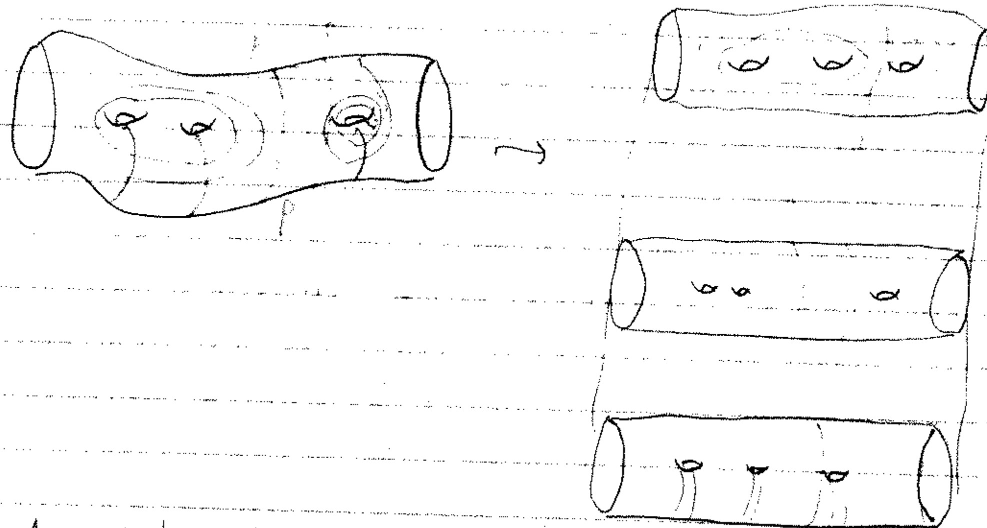
—— β-handle ———.

$f: M \to [-1, 4]$

$f^{-1}(-1) = R_-$

$f^{-1}(4) = R_+$

$$S(P) = P \underset{A \times \{\frac{3}{2}\}}{\cup} \left(A \times \overset{[\frac{3}{2}, 4]}{I} \right) \underset{B \times \{\frac{3}{2}\}}{\cup} \left(B \times \overset{[-1, \frac{3}{2}]}{I} \right)$$



Def. A product annulus is a properly embedded annulus $A \subset (M, \gamma)$ s.t. $A = S^1 \times [-1, 1]$,

$$S^1 \times \{-1\} \subseteq R_-$$
$$S^1 \times \{1\} \subseteq R_+ .$$

Note: The example above is a quasi-polygon whose associated surface $S(P)$ is a product annulus.

Thm. Let $(M, \gamma) \xrightarrow[\text{Annulus}]{\text{Product}} (M', \gamma')$ be a product annulus decomposition.
Then $SFH(M, \gamma) = SFH(M', \gamma')$
and (M, γ) product $\Leftrightarrow$ (M', γ') product.

Pf. Exercise.

(Note: Product disks & annuli are special because in an annulus or a bigon in Σ, the two circles/arcs of the boundary are isotopic.
So $A \cap \beta = \phi \Rightarrow A \cap \alpha = \phi$ and $B \cap \alpha = \phi \Rightarrow B \cap \beta = \phi$
$\Rightarrow$ Chain complexes computing $SFH(M, \gamma)$ & $SFH(M', \gamma')$ are identical.)

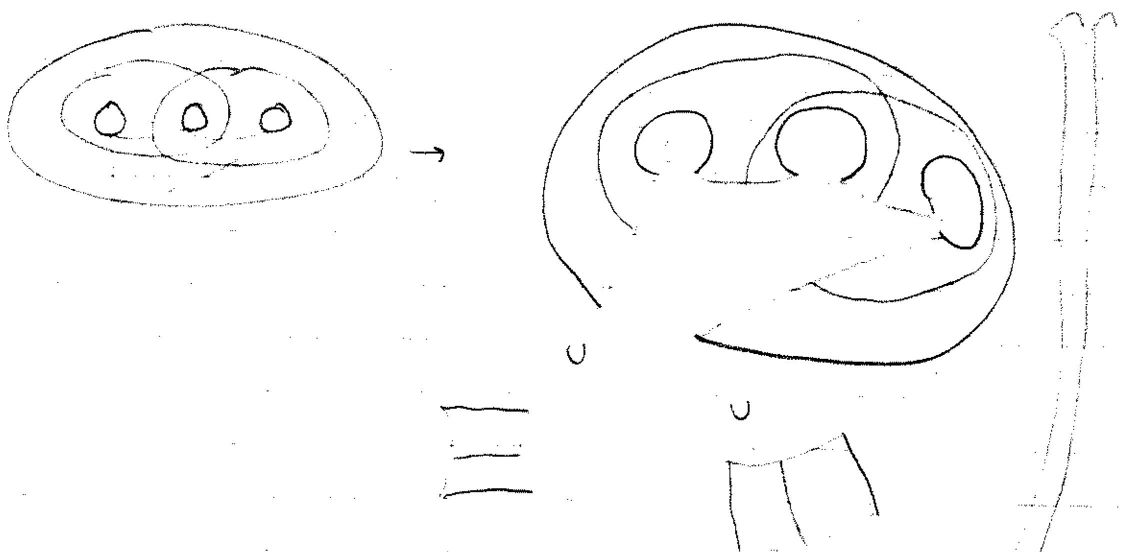
Not very strict conditions. The surface should intersect both R_+ & R_-. Probably one could simply relax the Def. of adapted H.D. to allow any surface.

Def. A H.D. for (M,γ) is adapted to S if (M,γ) has a quasi-polygon P s.t. $S(P) \propto S$.

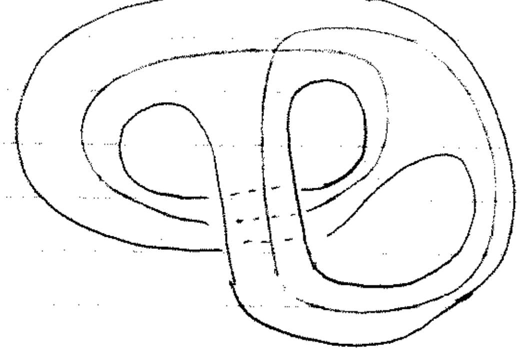
Lemma (A) "Any" decomposing surface has an adapted H.D.

(B) $(M,\gamma') \xrightarrow{S} (M',\gamma') \Rightarrow (M',\gamma')$ is associated to

$$\Big(\underset{\partial_B P \leftrightarrow \partial_B(\Sigma \setminus P)}{P_B \cup (\Sigma \setminus P)} \cup \underset{\partial_A P \leftrightarrow \partial_A(\Sigma \setminus P)}{P_A} =: \Sigma', \vec{\alpha}', \vec{\beta}'\Big)$$



Where P has $\vec{\alpha}$ & $\vec{\beta}$ arcs,
& $\vec{\alpha}', \vec{\beta}'$ are forced (by desire) to form closed curves.



$(\Sigma', \vec{\alpha}', \vec{\beta}')$

Pf. (A) Construct function explicitly on $\partial M \cup \text{nbhd}(S)$ & extend to (M,γ).
(B) Think about cut & paste def. of surface decomp., and (M,γ) resp. (M',γ').

Matt Hedden 4/28/11 HFH

Last Time:

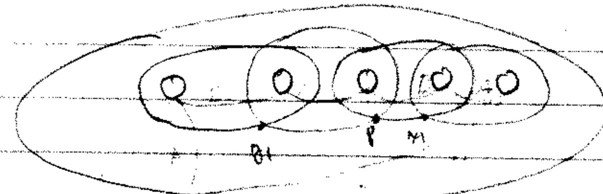
Def A H.D. adapted to a decomposing surface S

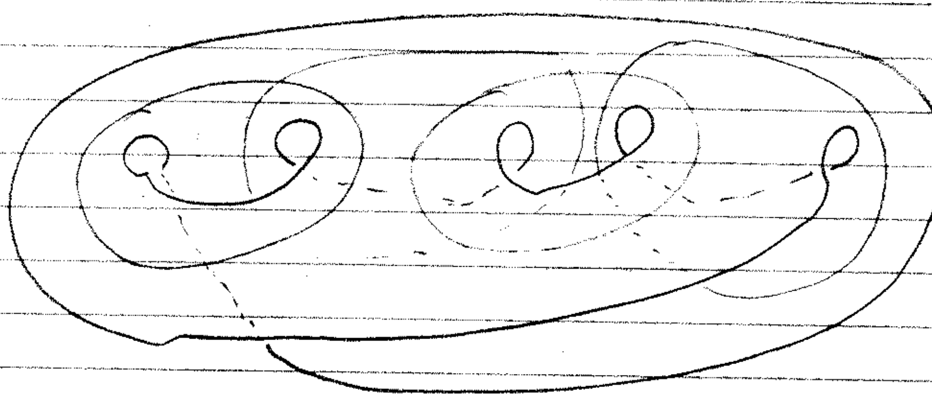
"majority of S lays on Heegaard surface Σ, is a quasi-polygon, 'P' "

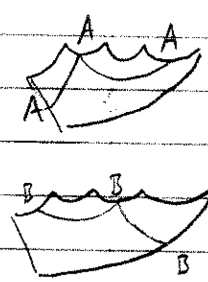
Def. $(\Sigma, \alpha, \beta; P) \xrightarrow{S(P)} (\Sigma', \alpha', \beta')$ Decomposing H.D.

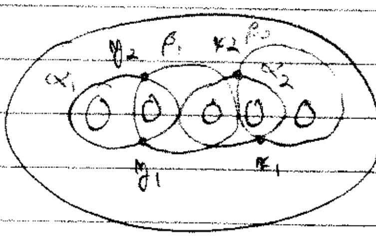
Lemma. Decomposed H.D. gives rise to $(M', \gamma') = (M, \gamma)$ decomposed along S.

Ex:









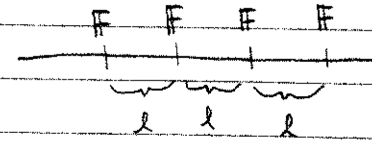
4 generators $\{y_i, x_j\}$ $i, j \in \{1, 2\}$

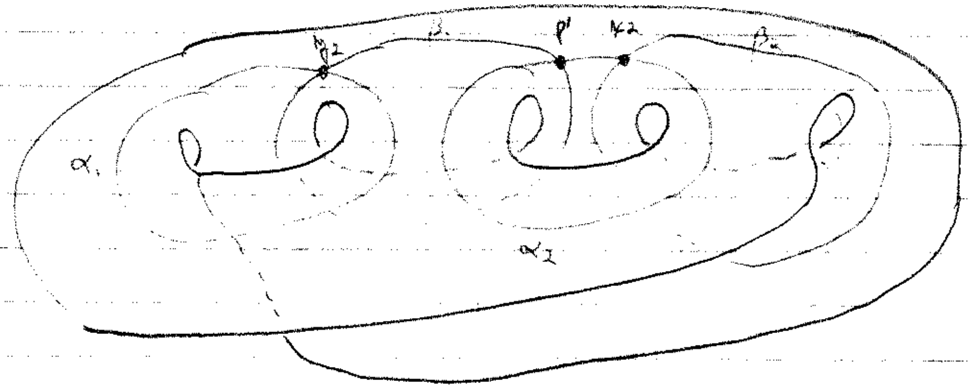
Explicit computation shows

$\epsilon(\{y_i x_j, y_\ell x_m\}) \neq 0$ unless $i = \ell$, $j = m$

Note: $H_1(M; \mathbb{Z}) \cong \mathbb{Z}$
$\underset{\epsilon(\bar{x}, \bar{y})}{\cup}$

$\Rightarrow SFH(M, \gamma) \cong \mathbb{F}^4$





<u>Note:</u> When we decompose the H.D., we lose any $\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ that has a component $x_i \in \vec{x}$ laying in P.

In this example, $\quad SFH(M', \gamma') = \mathbb{F}\langle \{x_2, y_2\} \rangle$

This example naturally generalizes to yield a map of groups

$$C(\Sigma', \alpha', \beta') \xhookrightarrow{i_P} C(\Sigma, \alpha, \beta, P)$$

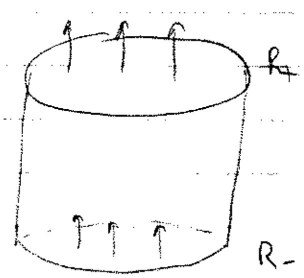
<u>Goal:</u> See that the map i_P is a chain map, and moreover, is an inclusion on homology.

i.e. $SFH(M', \gamma') \hookrightarrow SFH(M, \gamma)$

We need a more intrinsic splitting of $SFH(M, \gamma)$ than simply ε-classes.
We will use relative spin^c-structures.

<u>Def.</u> A <u>relative spin^c-structure on a sutured mnfld.</u> is a homology class of n.v.v.f. agreeing with v_{fix} on ∂M,

where $v_{fix} := \begin{cases} \text{inward normal on } R_- \\ \frac{\partial}{\partial t} \text{ on annulus } \gamma \times [-1, 1], \text{ nbhd (sutures)} \\ \text{outward normal on } R_+ \end{cases}$



Lemma. Let $\underline{s}, \underline{t} \in \mathrm{Spin}^c(M, \gamma)$. Then, $\exists$ a well-defined difference

$$\underline{s} - \underline{t} \in H^2(M, \partial M; \mathbb{Z})$$

Further, given $\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\exists$ map $\rightsquigarrow$ $\underline{s}(x) \in \mathrm{Spin}^c(M, \gamma)$.

(Pf. Modify $-\nabla f$ for a Morse function, as usual.)

And, $\underline{s}(x) - \underline{s}(y) = PD\left[\mathcal{E}(\vec{x}, \vec{y})\right]$.

Def. Call an $\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ outer wrt P if $x_i \notin P$ $\forall i = 1, \ldots, k = |\alpha| = |\beta|$.

Generators for $(\Sigma', \vec{\alpha}', \vec{\beta}')$ $\Leftrightarrow$ outer generators for $(\Sigma, \vec{\alpha}, \vec{\beta}, P)$.

$$\left(C(\text{outer}) = \bigoplus_{x \in \text{outer}} \mathbb{F}\langle x \rangle, \partial\right)$$

Claim. Outer generators generate a direct summand of $C(\Sigma, \vec{\alpha}, \vec{\beta})$.

Pf. It suffices to show that $\forall c \in C(\text{outer})$,

$$\partial c \subseteq C(\text{outer}) \quad \text{and}$$

$$\cancel{\forall c \in C(\Sigma, \vec{\alpha}, \vec{\beta}) / C(\text{outer}), \quad \partial c \subseteq C(\Sigma, \vec{\alpha}, \vec{\beta}) / C(\text{outer})}$$

~~This~~

$$\forall c \in C(\text{inner}), \quad \partial c \subseteq C(\text{inner}).$$

It would be E.T.S.

$$\mathcal{E}(x, y) \neq 0 \quad \forall x \in \text{inner}, \ y \in \text{outer}.$$

Pick $\vec{x}_{\text{inner}} \in \text{inner}$, $\vec{y}_{\text{outer}} \in \text{outer}$.

A representative for $\#_{\text{alg}} \mathcal{E}(\vec{x}_{\text{inner}}, \vec{y}_{\text{outer}}) \cap P \neq \emptyset$, which implies

$$\mathcal{E}(\vec{x}_{\text{inner}}, \vec{y}_{\text{outer}}) \neq 0 \in H_1(M, \gamma).$$

To see this, we can construct a representative for $\mathcal{E}(\vec{x}, \vec{y})$ by arcs

$$\left.\begin{array}{l} t_\alpha \quad \vec{x} \xrightarrow{\text{along } \alpha} \vec{y} \\ \cup \ t_\beta \quad \vec{y} \xrightarrow{\text{along } \beta} \vec{x} \end{array}\right\} \ t_\alpha, t_\beta \text{ are pushed slightly into their respective handlebodies.}$$

But $(t_\alpha \cup t_\beta) \cap P$ exactly at $x_i \in \vec{x}_{\text{inner}}$ lying in P.

Now we're done, since the orientation of $t_\alpha \cup t_\beta$ at $x_i \in \vec{x}_{\text{inner}} \cap P$ is always the same (either up or down)

So $\# \mathcal{E}(\vec{x}_{\text{inner}}, \vec{y}_{\text{outer}}) \cap [S(P)] \neq 0$.

We've shown that

$$\begin{array}{c} C(\text{outer}) \\ \text{||2} \leftarrow \text{group iso.} \\ C(\Sigma', \vec{\alpha}', \vec{\beta}') \end{array} \quad \& \quad (C(\text{outer}), \partial) \underset{\text{Direct summand}}{\triangleleft} C(\Sigma, \vec{\alpha}, \vec{\beta}).$$

Need to see that $\partial^{\Sigma} = \partial^{\Sigma'}$

$\partial^{\Sigma'}$ ← Differential on $C(\Sigma', \vec{\alpha}', \vec{\beta}')$.

∂^{Σ} ← Differential on $C(\text{outer})$

Juhasz' proof that $\partial^{\Sigma} = \partial^{\Sigma'}$ goes by way of "nice" diagrams

i.e. H.D. where all regions of $\Sigma \setminus \{\vec{\alpha} \cup \vec{\beta}\}$ not adjacent to $\partial\Sigma$ are bigons or rectangles.
When a H.D. is nice, the differential only counts $\phi \in \pi_2(\vec{x}, \vec{y})$ whose domains are embedded bigons or rectangles (which contribute ± 1 by RMT.)

Juhasz refines the Sarkar-Wang algorithm to find a nice diagram so that
(A) $(\Sigma, \vec{\alpha}, \vec{\beta}, P)$ is nice
(B) $(\Sigma', \vec{\alpha}', \vec{\beta}')$ is nice
Then, observes that $\exists$ bijection between domains $\phi \in \pi_2(\vec{x}, \vec{y})$ $\vec{x}, \vec{y} \in \text{outer}$
and $\phi' \in \pi_2(\vec{x}', \vec{y}')$ $\vec{x}', \vec{y}' \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$.

$K \subseteq Y$.

<u>Thm.</u> $g(K) = \max\{ i \mid \widehat{HFK}(K, i) \neq 0 \}$.

<u>Pf.</u> Let F be a min. genus Seifert surface for K

$$(Y \setminus \text{nhd.}(K), \gamma_2) \xrightarrow{F} (Y \setminus F, \partial F) \xrightarrow{\text{Gabai's hierarchy}} (R \times I, \partial R \times I)$$

γ_2 ← meridians

$\oplus$ SFH(i S) = $SFH(Y \setminus nb(K), \gamma)$ $\xrightarrow{\text{By previous Thm.}}$ $SFH(Y \setminus F, \partial F) \longleftrightarrow SFH(R \times I, \partial R \times I)$

inc( )

$SFH(Y \setminus nb(K), \gamma) \cong \widehat{HFK}(Y, K)$, $SFH(R \times I, \partial R \times I) \cong \mathbb{F}$

By previous Thm, this inclusion $(i_F)_*$ maps into $C(\text{outer}) \cong \widehat{HFK}(Y, K, g(F))$

This non-vanishing & Adjunction Inequality (i.e. $\widehat{HFK}(Y, K, i) = 0$ if $i > g(K)$) yields the Thm. $\square$

Alexander grading
$\frac{1}{2}\left(\langle c_1(\xi), [F, \partial]\rangle + PD([\lambda]), [F, \partial]\right)$