Let \[ Y^3 \] be a 3-manifold. Define the \( \text{HF}^0(Y, \pm) \) Floer homology groups. Consider the map \( w^0 \) that associates to each \( Y^3 \) a \( \text{HF}^0(Y, \epsilon_Y) \) group. The key theorem is:

\[ \text{Key} Y^3 \to \bigoplus_{i \in \mathbb{Z}} \text{HFK}(Y, k, i) \]

Theorem (0.5) \[ \text{Key} S^3, g(K) = \max \left\{ i \in \mathbb{Z} : \text{HFK}(S^3, K, i) \neq 0 \right\} \]

Definition: A knot \( K \subset S^3 \) is fibered if \( S^3 - n(K) \) is a fiber bundle over \( \Sigma^2 \), with 2-dimensional fiber.

Example: Unknot

\[ D^2 \subset S^3 - n(U) \cong D^2 \times S^1 \]

Example: Show that \( U \) is trivially fibered.
Prop: If \( K \) is fibered, then \( \Delta_K(T) \) is monic.

\[ \Delta(T) = -2T + 5 - 2T^{-1}. \]

Thm (0.5) If \( K \) is fibered, then \( \text{rank}(\text{HF}(K, \gamma(K))) = 1 \).

\[ (N, H_1(N), H_2(N), \text{rank}(\text{HF}(K, \gamma(K)))) = 1 \implies K \text{ is fibered}. \]

Thurston Norm

\( Y \) is a closed, orientable, 3-manifold, \( \Sigma \in H_2(Y; \mathbb{Z}) \).

Def: \( \Sigma \) is realized by an embedded surface \( \Sigma \subseteq Y \) if

\[ i_\#([\Sigma]) = \Sigma \in H_2(Y; \mathbb{Z}). \]

Exercise: Show that any \( \Sigma \in H_2(Y) \) is realized by an embedded surface.

\[ \text{Def: Let } \chi_-(\Sigma) = \sum \max \left\{ -\chi(\Sigma), 0 \right\}, \]

\[ \Sigma \subseteq Y, \chi(\Sigma) \in \mathbb{Z} \]

\[ \text{Def: } \Theta : H_2(Y; \mathbb{Z}) \to \mathbb{Z}^{\geq 0} \text{ is the Thurston semi-norm, } \]

\[ \Theta(\Sigma) = \min \left\{ \chi_-(\Sigma) : \Sigma \text{ realizes } \Sigma \right\}. \]

\( \Sigma \subseteq \text{Spin}^c(Y) \) is an oriented 2-dimensional bordism.

\[ \Sigma \]

\( c_1(\Sigma) \in H^2(Y) \) is the 1st Chern class of \( \Sigma \).
Thm (0.5) \( S \in H_2(\Sigma) \).
\[ \Theta(S) = \max \{ \langle c_1(\Sigma), S \rangle | S \in \Sigma_{2g}^+ \cap c_1(\Sigma), \hat{HF}(\Sigma, S) \neq 0 \} \]

Gist: Floer homology solves the minimal genus problem for 3-manifolds (possibly with torus boundary)

What about dim 4?

Def. \( q_4(K) = \min \{ \text{genus}(Z) : i : \Sigma \subset B^4, \text{smoothly embedded} \} \)

called a smooth 4-ball genus, or 4-slice genus.

Exercise: \( q_4(K) \leq q(K) \).

Def. \( u(K) = \min \# \text{crossings necessary to change in any projection of } K \) to unknot it.

Ex.: \( u(\begin{tikzpicture} [scale=0.5]
  \draw (-1,0) to[bend right] (1,0);
  \draw (-1,0) to[bend left] (1,0);
\end{tikzpicture}) = 1 \) because \( \begin{tikzpicture} [scale=0.5]
  \draw (-1,0) to[bend right] (1,0);
  \draw (-1,0) to[bend left] (1,0);
\end{tikzpicture} \) is a projection of \( K \) unknot.

Exercise: \( q_4(K) \leq u(K) \)

Hint: The unknotting transformation gives a move.

Use this to construct a surface in \( B^4 \).

Floor homology

\[ K \rightarrow \widehat{HF}(K) \]

\[ c(K) \in \mathbb{Z} \]

Thm (0.5, Kroner)

\[ |c(K)| \leq q_4(K) \]
Def. A **torus knot** \( T_{p,q} \) is a knot which embeds in a torus of slope \( \frac{p}{q} \).

\[ \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} = \mathbb{T}^2. \]

\[ \{z \in \mathbb{C}^2 \mid z^p + w^q = 0\} \cap \{z \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} = \mathbb{S}^3 \]

**Exercise:** Show that \( K_{p,q} \) is a knot if \( (p,q) = 1 \).

- Show that \( K_{p,q} \) is a torus knot \( T_{p,q} \).
- Show that \( \chi(V_{p,q} \cap \{|x|^2 + |y|^2 \leq 1\}) = pq - p - q \).

**Hint:** Riemann-Hurwitz formula.

\[ \chi(\Omega) = 1 - 2g(V_{p,q}) - \frac{1}{2} - q - q. \]

\[ \Rightarrow \chi(V_{p,q}) = \frac{(p-1)(q-1)}{2}. \]

Milnor Conjecture: \( \eta(T_{p,q}) = \frac{(p-1)(q-1)}{2} = \chi(T_{p,q}) \)

**Theorem (Kronheimer, Milnor):** \( 0 < R \)

\[ \eta(T_{p,q}) = \frac{(p-1)(q-1)}{2} \leq \frac{p+q}{2} \]

\[ \chi(T_{p,q}) \leq \frac{(p+q)(p+q-1)}{2}. \]

**Corollary (Habli):**

Milnor Conjecture: \( \exists q \in \mathbb{Q} \) such that \( X \) is homeomorphic to \( \mathbb{R}^4 \)

but not diffeomorphic to \( \mathbb{R}^4 \).

\( (\chi \in \text{ExtH}_{\mathbb{Q}} \text{ of } \mathbb{R}^4) \)
Dehn Surgery

Def: A lens space \( L(p, q) \) is the manifold obtained by \(-\frac{p}{q}\) on \( \mathbb{S}^3 \).
(Note: OS choose for opposite orientation here.)

Q: Which knots can I do Dehn surgery on and obtain a lens space?

Thm. Suppose \( K \subseteq S^3 \) admits a lens space surgery. (i.e. \( S^3/K \approx L(p, q) \) for \( p > 0, (r, s) = 1 \)).

Then,
1. (0.5) All coefficients of \( \Delta_K(T) \) are \( \pm 1 \).
2. (0.5) \( q(K) = q(L) \).
3. (Neu/Choi/Snek) \( K \) is fibered.
4. (Hidden) \( K \) bounds a rational curve \( V_K \subseteq B^4 \).

(At least the last 3 have no known proofs that do not involve HFH technology.)

\( \overleftarrow{\text{HFK}} \) could possibly give a classification of such knots.

*(Because) Determined which \( L(p, q) \) you p+ by doing surgery on \( K \neq \text{unknot} \).

Thm. (0.5) Let \( L \subseteq S^3 \) be an alternating link, and let \( \Sigma_2(L) \) be its branched double cover.

Then, \( \Sigma_2(L) \cong (X^4, \omega) \) s.t. \( X^4 = S^3 \odot \Sigma_2(L) \) is symplectic.

Also: Applications to

- Floer Theory
- Contact geometry
- Concordance = homology cobordism
- \( \pm \text{ contact form} \)

\( V_{\text{class}} \text{ to knots into} \) \( \Sigma_2(L) \)

E.g. lens spaces
Plant: 1. Morse homology
   (2) Lagrangian Floer homology
   - Milnor "Morse Theory" p. 1-27, 32-38
   - Gompf-Stepsic \( "\text{4-manifolds \& Kirby Calculus}" \) p. 69-82
   - Hutchings (Prelim) with S. Bublyshko-Hutchings "Lectures on Morse homology" p. (?)
   - McDuFF "Floer Theory \& low dim. topology"

Recall: A smooth function \( f: M \to \mathbb{R} \) on a manifold \( M \) is Morse if critical pts. are isolated and have a local form:
\[
f = -\lambda_1^2 - \lambda_2^2 - \cdots - \lambda_k^2 + \lambda_{k+1}^2 + \cdots + \lambda_n^2,
\]
\( k \) is called the index of the critical pt.

If \( f^{-1}([-a, b]) \) doesn't contain a critical pt., then \( f^{-1}(a) \) and \( f^{-1}(b) \) are diffeo,
and \( f^{-1}(\pm a) \) and \( f^{-1}(\pm b) \) are diffeo.

\[\Phi : M \times \mathbb{R} \to M \text{ is the flow of } -\nabla f\]

\[\text{i.e. the vector field s.t. } \Phi (-\nabla f, -) = -df \]

\[\Phi \text{ is a Riemann metric } k\text{-hundle}\]

Then, if \( \lambda \) is a critical value of \( f \), then
\[f^{-1}((\pm \lambda - \varepsilon)] \cup (0^k \times 0^{n-k}) = f^{-1}((\pm \lambda + \varepsilon))\]

when \( n = \dim(M \times \mathbb{R}) \), \( k \) = index of \( f \) at \( p \), \( f(p) = \lambda \).

\[\text{Ex: } \]

\[\begin{align*}
\begin{array}{c}
\text{Diagram 1:} \\
\text{Diagram 2:}
\end{array}
\end{align*}\]
Morse homology

\[ \text{Crit}(f) = \{ \text{critical pts. of } f \} \quad p \in \text{Crit}(f). \]

\[ W^u(p) = \{ x \in M \mid \lim_{t \to +\infty} \varphi(x, t) = p \} \]

\[ W^s(p) = \{ x \in M \mid \lim_{t \to -\infty} \varphi(x, t) = p \} \]

**Def.** Given \( p, q \in \text{Crit}(f) \), call a curve \( \gamma(t) \) a gradient flow connecting \( p \) to \( q \) if

\[ \lim_{t \to +\infty} \gamma(t) = p \quad \text{and} \quad \lim_{t \to -\infty} \gamma(t) = q \]

and

\[ \frac{d\gamma}{dt} = -\nabla f(\gamma(t)) \quad t \to +\infty \]

**Lemma.** Gradient flows connecting \( p \) to \( q \) \( \iff \) \( W^u(p) \cap W^s(q) \).

**Prop.** \( W^u(p) \) is a smooth submanifold of \( M \) diffeomorphic to an open ball of dimension \( \text{ind}(p) \),

and \( W^s(p) \) is a smooth submanifold of dimension \( \text{dim}(M) - \text{ind}(p) \).

**Not.** If \( W^u(p) \cap W^s(q) \), then \( W^u(p) \cap W^s(q) \) is a smooth manifold of dimension \( \text{ind}(p) - \text{ind}(q) \).

(2) If \( \gamma(t) \) is a gradient flow from \( p \) to \( q \), then \( \gamma_c(t) = \gamma(t + c) \) is also a gradient flow from \( p \) to \( q \).

\[ W^u(p) \cap W^s(q) \times \mathbb{R} \to W^u(p) \cap W^s(q) \]

\[ (\gamma(t), c) \mapsto (\gamma(t + c)) \]

is a free \( \mathbb{R} \)-action provided \( \gamma(t) \neq \text{constant} \).

\[ M(p,q) := \frac{W^u(p) \cap W^s(q)}{\mathbb{R}} \]

"Morse space of gradient flows"

set of unparameterized gradient flow curves from \( p \) to \( q \).

\( M(p,q) \) is a manifold of dimension \( \text{ind}(p) - \text{ind}(q) - 1 \).

**Def.** \( \text{Crit}_i(f) = \{ \text{critical } \text{pts. } i \} \)

\( \text{index } i \text{ critical pts. } i \cdot \)

\[ C_*(f) : = \bigoplus_{i \in \text{Crit}_i(f)} \mathbb{Z}/2 \langle p \rangle \]

metric

\[ d : C_* \to C_{*-1} \quad \text{def.} \quad d(p) = \sum_{f \in \text{Crit}_{i-1}(f)} \# M(p,q) q \]
\[ \text{Thm.} \quad 2^2 = 0, \]
\[ \text{Pf.} \quad \delta \circ \partial (f) = \partial \left( \sum_{\mathcal{C} \in \mathcal{C}_{\text{Crit}} - 1} \varepsilon \cdot \mathcal{H}(a, f) \cdot q \right) \]
\[ = \sum_{\mathcal{C} \in \mathcal{C}_{\text{Crit}} - 2} \left( \sum_{\mathcal{C} \in \mathcal{C}_{\text{Crit}} - 1} \varepsilon \cdot \mathcal{H}(a, f) \cdot q \right) \cdot \mathcal{H}(a, f) \cdot r \]
\[ \text{(8)} \quad 0 = \sum_{\mathcal{C} \in \mathcal{C}_{\text{Crit}} - 2} \varepsilon \cdot \mathcal{H}(a, f) \cdot \mathcal{H}(a, f) \quad \forall \mathcal{C} \in \mathcal{C}_{\text{Crit}} - 2. \]

Why is (8) True?
\[ \dim \mathcal{H}(a, f) = 1 \quad \text{i.e.} \quad \mathcal{H}(a, f) = \{ 1 \}\text{-dim line field} \]
\[ \mathcal{H}(a, f) \text{ has a natural compactification obtained by adding } "\text{broken gradient flow lines}" \]
\[ \text{from } p \text{ to } r. \]

A broken flow line is a pair of flow lines connecting \( p \) to \( q \) and \( q \) to \( r \) respectively.

\[ \# \text{ boundary pts. of } \mathcal{H}(a, f) = 0 \text{ and } 2 \]
\[ \text{and these boundary pts. correspond to broken flow lines.} \]
\[ \text{which are exactly counted by the product } \# \mathcal{H}(a, f) \cdot \mathcal{H}(a, f). \]
\[ \mathcal{H}(a, f) \text{ is compactified by broken flow line "completion"} \]
\[ \text{Every break, flow line is added under this compactification.} \]

Lagrangian Floer Homology
\[ \text{Want to do the same thing on an infinite dim'l space } \Omega(L, L). \]
\[ \text{space of paths connecting two submanifolds } L_0 \text{ to } L_1. \]

Motivation: get bounds for \( \# \) geometric intersections of submanifolds and complicatory dimension.

Possible if we have a Morse homology for a function
\[ \text{on } \Omega(L_0, L_1) \text{ whose critical points are constant paths.} \]
Context: Symplectic geometry.

Let \((M^{2n}, \omega)\) be a symplectic manifold.

\[ \omega = \text{closed 2-form} \quad (d\omega = 0, \quad \omega \in \Omega^2(M)) \]

Nondegenerate \(\omega \Rightarrow \omega > 0\)

Def. Let \(C(M; L_0, L_1) := \bigoplus_{L_0 \cap L_1} \mathbb{R}/2\langle \delta \rangle \)

What is \(\alpha\) Morse function? There isn't one. But we don't need as many here.

Really, \(\mathcal{J}(\Delta f, -) = df\), and critical pts. are \((0, \partial\alpha)\).

So, maybe it suffices to have a 1-form, and some notion of a metric.

Def. \(\alpha : T\Omega(L_0, L_1) \to \mathbb{R}\)

\[
\alpha Y(s) = \int_0^1 \omega(Y(s), \dot{Y}(s))\, ds
\]

\[ u(s, t) : [0, 1] \times [-1, 1] \to M \]

\[ u(s, 0) = y(s) \quad u(1, t) \in L_1. \]

\[ \frac{du(s, t)}{dt} \bigg|_{(s, t)} \]

\[ T\Omega(L_0, L_1) = \{ \text{paths}, Y(s) \in T_{y(s)}M \} \]

Exercise: Show that \(\alpha \equiv 0 \iff Y(s) \text{ constant} \iff Y(s) \in L_0 \cap L_1.\)
Last Time:

Morse homology

\[ M^n, f \rightarrow H_n \quad g \rightarrow C(f) = \bigoplus_{x \in M^n} \mathbb{Z}/2 \mathbb{Z} \langle x \rangle \]

\[ d \text{ counts gradient flows connecting } x \text{ to } y \text{ if } \text{ind}(x) - \text{ind}(y) = 1. \]

Legendrian Floer homology

\[ (L^2, \omega) \rightarrow \Omega^ \infty (L_0, L_1) \]

\[ \alpha : \Omega^ \infty (L_0, L_1) \rightarrow \mathbb{R} \]

\[ \alpha_\varphi (\xi(s)) = \int_{s_0}^{s_1} \omega \left( \frac{d}{ds} (s'), \xi(s') \right) ds'. \]

\[ C(L_0, L_1) = \bigoplus_{x \in L_0, L_1} \mathbb{Z}/2 \mathbb{Z} \langle x \rangle \]

What does \( d \) count?

Recall:

\[ g(-\nabla f, -) = df \]

Metric on \( \Omega^ \infty \):

\[ g_{\varphi} (\xi(s), \eta(s')) = ? \]

Def. An almost complex structure is a bundle automorphism

\[ J : TM^{2n} \rightarrow TM^{2n} \quad \text{s.t. } J^2 = -\text{Id.} \]

Def. Call \( J \) an a.c.s. on \( (M, \omega) \) compatible with \( \omega \) if

- \( \omega(Jv, v) > 0 \) if \( v \neq 0 \)
- \( \omega(Jv, Jw) = \omega(v, w) \) \( \forall v, w. \)

Proposition (See McDuff-Salamon, Intro Book)

Compatible almost complex structures on \( M \) for any symplectic manifold \( M \) and the space of compatible a.c.s. for a given form is contractible.
$g^{\gamma \delta}(\gamma, \delta) = \int_0^1 \omega_{\nu(s)}(J \delta(s'), \gamma(s')) \, ds'\nabla_{\nu(s)}(\gamma, \delta) = \alpha_\gamma(\delta)$

Trying to find what this gradient vector field is:

This equation reduces to:

$$\int_0^1 \omega_{\nu(s)}(J(-\text{grad}\chi(s')) \gamma(s')) \, ds' = \int_0^1 \omega_{\nu(s)}(\frac{d\gamma}{ds}(s'), J(s')) \, ds'$$

$\Rightarrow \quad \frac{d\gamma}{ds}(s') = J(-\text{grad}\chi(s'))$

Applying $J$ to both sides:

$J\left(\frac{d\gamma}{dt}(s')\right) = \text{grad}\chi(s')$

Note: $J_{\nu(s)}$ usually changes with $\nu$ in the manifold as well.

Want $\partial_t$ to count gradient flows connecting critical points $\gamma$ to $\nu$.

$u(s, t) : [0, 1] \times \mathbb{R}$

$$\frac{du}{dt}(s', t') = -\text{grad} u(s', t') = -J_{\nu(s', t')} \frac{du}{ds}(s', t')$$

Apply $J$ again:

$$J\left(\frac{du}{dt}(s', t')\right) = \frac{du}{ds}(s', t') \quad (*)$$

$(*)$ is the $J$-holomorphic curve equation.

Exercise: $(*) \iff J \circ du = du \circ i$

When $\iota$ is the (almost) complex structure on $[0, 1] \times \mathbb{R} \subseteq (\mathbb{C}, \iota)$.

Note: $(\#)$ means that $\nu$ is a map into a complex plane on the tangent bundle.
Exercise: Show \( (*) \) is equivalent to the Cauchy–Riemann equations
\[
\omega = \partial_x \wedge \partial_y
\]

Def. \( M(x, y) := \{ \mathbf{u} = [0, 1] \times \mathbb{R} \rightarrow M^2 \mid \mathbf{u}(t, 0) \in L_0, \ \mathbf{u}(1, t) \in L_1, \ x = y \text{ if } t = 0, \ y = y \text{ if } t = 1 \} \)
\[
\mathbf{d} \omega \circ \mathbf{u} = J \circ \mathbf{d} \mathbf{u}
\]

Convention:

- Space of gradient trajectories (flow lines)
- Space of \( J \)-holomorphic disks (strips) connecting \( x \) to \( y \).
  (pseudo-holomorphic)

Note: "Disk" is justified since \([0, 1] \times \mathbb{R}\) is conformally equivalent by Riemann mapping theorem.

Define \( \hat{d} \mathcal{M} = \sum_{\gamma \in \mathcal{L}_1} \hat{\mathcal{M}}(x, y) \cdot \gamma \)

\( \hat{\mathcal{M}}(x, y) \) of \( J \)-holomorphic disks connecting \( x \) to \( y \) (and \( \Sigma \)).

Why is this number \( \hat{\mathcal{M}}(x, y) \) well-defined?

For Morse theory, \( \# \mathcal{M}(x, y) \) was well-defined provided \( \text{ind}(x) - \text{ind}(y) = 1 \).

What plays the role of index? Something called Maslov index.

Is \( \hat{\mathcal{M}}(x, y) \) a manifold (of any dimension)?
We need some sort of transversality assumption to ensure $\mathcal{M}(x, y)$ is a manifold. For Morse theory, we could do this, because of Sard's theorem.

There is an infinite diml. version of Sard's theorem which allows this.

(Nothing more will be said about this).

Is $\# \mathcal{M}(x, y)$ finite? If so, then by fixed study plus transversality, tells us we have a smooth manifold of dimension $0$. (We don't have compactness)

Let notice: $\frac{d u(s, t)}{d s} = -\frac{\partial u(s, t)}{\partial s}$ is invariant under $u(s, t) \rightarrow u(s, t + \epsilon) = u_c(s, t)$

So $\tilde{\mathcal{M}}(x, y) = \mathcal{M}(x, y) / \mathbb{R}$

Fininess is ensured, one hopes by some compactness theorem for $\mathcal{M}(x, y)$.

"Gromov compactness"

$\mathcal{A} = 0$. This will hold because of Gromov compactness and a Gluing Theorem.

Thm. (Floer)

Let $(M^2, L_0, L_1)$ be a symplectic manifold and two Lagrangian submanifolds.


1. $L_0 \# \# L_1$

2. $M$ is compact

3. $\pi_2(M^2, \mathbb{R}) = \pi_2(M, L_0) = \pi_2(M, L_1) = 0$.

Then $\mathcal{A} = 0$ and $H_\times(\mathcal{C}(L_0, L_1), \mathcal{A})$ depends only on $M$ up to symplectomorphism and $L_1$ up to Hamiltonian isotopy.

How can we use this construction to study $3$-manifolds?

Could try $(Y^3 \times \mathbb{R})^+$ to give our diml. symplectic manifold.

This is called Embedded Contact homology (ECH).

Hutchings, Taubes

Amazingly, this gives an invariant of $3$-manifolds isomorphic to our charge.
Heegaard Floer Homology

What is a 3-manifold, anyway?

To see them, we'll use a device called Heegaard diagrams.

Def. A Heegaard diagram is a 3-tuple \((\Sigma, \alpha, \beta)\) where:

1. \(\Sigma\) is a closed, oriented surface of genus \(g\).
2. \(\alpha = \{\alpha_1, \ldots, \alpha_g\}\) is a collection of \(g\) s.c.c. in \(\Sigma\), pairwise disjoint, and \(\text{span}(\{\alpha_i\}) \leq 4(\Sigma, \beta)\) is \(g\) dimensional.
3. \(\beta = \{\beta_1, \ldots, \beta_g\}\)

Prop. A Heegaard diagram specifies a unique (up to homeomorphism) closed orientable 3-manifold.

- Need Heegaard moves to go between 2 pictures for some 3-manifl.
Recall the def. of a Heegaard diagram. We will also require $\alpha \cap \beta$.

Prop. Any HD specifies a unique, oriented homeomorphism class of closed 3-manifolds, $Y$.

Pf.

\[ \Sigma \times I \xrightarrow{\phi} \Sigma \times I \]

Attach a 2-handle to $\Sigma \times E^3$, one for each $\alpha$ curve.

Note: Each $\times$ curve has a nbhd. homeomorphic to an annulus $S^1 \times \{ \alpha \} \cong \text{nbhd}(\alpha) \subseteq \Sigma \times E^3$.

Form $\Sigma$ space $\Sigma \times I \cup \{ \beta \text{ 2-handle} \}$.

Exercise: $\partial_+ Y_{\text{disks}} \cong S^2 \cong \partial Y_{\text{disks}}$.

Hint: This follows from the condition that $\Sigma(\beta)$ spans a $q$-dim subspace of $H_1(\Sigma \times E^3)$ ($H_1(\Sigma \times E^3)$).

Glue $E^3$ to $\partial_+ Y_{\text{disks}}$ along $\partial E^3$.

This gives a closed 3-manifold.

Uniqueness follows from the diffeomorphism allowed in the handle-attachment and knowing that there are unique.

Thm. Any 3-manifold can be given a HD.

We (Idea): A HD comes from a special type of Morse function on a 3-manifold.

1. $f: Y^3 \to \mathbb{R}$, Morse

2. $f$ is self-indexing, i.e., $\text{ind}_f(p) = f(p)$ \( \forall p \in C_{\text{crit}}(f) \).

(N.B. This implies $f_{\text{crit}}(Y) \subseteq [0, 3]$)

3. $f$ has a unique index 0 and unique index 3 critical pt.

Assuming such a function, we use it to construct a HD.
\[ \Sigma = f^{-1}(\{ \frac{3}{2} \}). \]

\[ \frac{3}{2} \] is a regular value of \( f \). 5, \( f^{-1}(\frac{3}{2}) \) is a 2-dim submanifold.

Claim: \( \# \{ \text{index 1 critical pts} \} = \# \{ \text{index 2 critical pts} \} \)

Proof: Follows from the fact that there are unique index 0 and index 3 critical pts.

Our diagram will be:

\[ (\Sigma = f^{-1}(\{ \frac{3}{2} \}), \quad \Sigma^2 = W^s(\text{ind. 1 crit. pts}) \cap \Sigma, \quad \Sigma^3 = W^u(\text{ind. 2 crit. pts}) \cap \Sigma) \]

Alternatively, look at the concaves and convexes of the 1-boundary and 2-boundary resp.

Exercise 1: Think about what a HD for a 2-manifold \( \Phi \) should look like.

Exercise 2: What types of Morse functions give rise to such diagrams?

Warning: Draw a HD for the compliment of a trefoil knot.

What about the special Morse functions?

1. Morse functions exist (by Sard’s theorem) \( \text{Morse(M,R)} \leq \text{Maps(M,R)} \).
2. Self-indexing Morse functions exist (see Ch. 4 of Milnor “Lectures on the h-Cobordism Theorem”)
3. The simple index 0 and 3 critical pts. require a Hille’s Cancellation Lemma (Milnor)

\[ (\text{descending (p)} \cap \text{ascending (p) }) / R = \text{Ent. 3} \]

They mean “circular” \( f \) and \( f \), i.e., find a family of functions \( f_s \) s.t.

\[ f_0 = f, \quad f_f = f \] is a function w/o critical pts. \( p \) \& \( q \), having all other crit pts.

clue
Theorem. Any two HDs for the same 3-manifold can be connected by a sequence of moves:

1. Stabilization or its inverse (destabilization).
2. Handle slides of \( \alpha \) curves over \( \alpha \) (or \( \beta \) curves over \( \beta \)).

- Handle slide of \( \alpha_1 \) over \( \alpha_2 \).

\( \beta \), \( \beta_0 \) and \( \beta_2 \) are not homotopically independent, so \( \beta_0, \beta_2 \) can be related to \( \beta \).
Ian of PE.

$$\begin{array}{c}
\gamma \in \gamma^0 \left( \mathbb{R}, \mathbb{R}^3 \right)
\end{array}$$

$$\gamma(0) = f_0$$,
$$\gamma(1) = f_1$$.

At only finitely many pts. in this path do the functions not Morse.

Understanding what can happen there gives us the 3-manifold.

Theorem (Morse Homology)

Try:

$$\begin{array}{c}
\left( \mathbb{C}, \mathbb{C} \times \mathbb{C} \right) \to \mathbb{C}(\mathbb{C}, \mathbb{C}) = \bigoplus_{n \in \mathbb{Z}} \mathbb{Z}/2 \langle x \rangle
\end{array}$$

The $\partial$ is the gradient of $\nabla$.

$$\begin{array}{c}
\mathbb{Z}/2 \langle x \rangle \quad \nabla
\end{array}$$

$$\begin{array}{c}
\delta \nabla = 0. \quad \text{except the constant term}.
\end{array}$$

$$\begin{array}{c}
\mathbb{Z}/2 \langle x \rangle \oplus \mathbb{Z}/2 \langle y \rangle
\end{array}$$

$$\begin{array}{c}
\delta \nabla = 0, \quad \delta y = 0
\end{array}$$

$$\Rightarrow H_\ast \left( \mathbb{C}(\mathbb{C}, \mathbb{C}), \partial \right) \cong \mathbb{Z}/2$$

So this is a invariant under the moves.

That's not to say it's not interesting - just that it doesn't give us a manifold invariant.

Here's what we do:

$$\begin{array}{c}
\left( \Sigma, \omega, \beta \right) \to \text{Sym} \beta (\Sigma) = \mathbb{C} \times \cdots \times \mathbb{C} / S_{\Sigma}
\end{array}$$

$$\text{Sym} \beta (\Sigma)$$ is smooth, complex, and symplectic!

Pf: Smith = complex follow from FTA! $$\left( \text{Sym} \beta (\Sigma) \cong \mathbb{C} \right)$$
\[ \pi_0 := \alpha \times \alpha_2 \times \ldots \times \alpha_g / S_g \]
\[ \pi_0 := \beta_1 \times \beta_2 \times \ldots \times \beta_g / S_g \]

\[ C(\Sigma, \pi_0) = \bigoplus \mathbb{Z}/2 \langle \alpha \rangle \]

\[ \forall \pi_0 \land \pi_0 \in \text{sym} \mathfrak{E} \]

\( \delta \) counts J-holomorphic disks connecting \( \alpha \) to \( \beta \).

**Thm. (0.5)** The homology of \( C_*(\Sigma, \pi_0) \) is a 3-manifold invariant. i.e., it doesn't depend on the diagram.

**Thm. (0.5)** \( \dim H^3_*(Y^3) = \text{order} | H_1(Y) | \)

if \( H_3(Y; \mathbb{R}) \) is finite.