LOCALLY FINITE SIMPLE GROUPS OF FINITARY LINEAR TRANSFORMATIONS

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Abstract. The classification of the groups of the title is a natural successor to the classifications of finite simple groups and of locally finite linear simple groups. The statement of the classification is given along with an explanation of the examples. The proof is then discussed.

Key words: Simple group, locally finite group, finitary linear group

1. Introduction

Let ${}_K V$ be a left vector space over the field K. The element $g \in GL_K(V)$ is *finitary* if $V(g-1) = [V, g]$ has finite K-dimension. This dimension is the *degree* of g on V, $\deg_V g = \dim_K [V, g]$. The invertible finitary linear transformations of V form a normal subgroup $FGL_K(V)$ of $GL_K(V)$, and any subgroup of $FGL_K(V)$ is called a *finitary linear group*, obvious examples being the linear groups, subgroups of $GL_K(V)$ for finite dimensional V. We stretch this terminology further by referring to any group which is isomorphic to a finitary linear group as a *finitary group.* Similarly, a *linear group* is any group which is isomorphic to a subgroup of $GL_K(V)$, for some finite dimensional V.

The classification of locally finite simple groups of finitary linear transformations to be discussed here is a natural successor to **CFSG**, the classification of finite simple groups [**?**], and the work **BBHST** of Belyaev [**?**], Borovik [**?**], Hartley and Shute [**?**], and Thomas [**?**] which classified the locally finite simple groups that are linear.

(1.1) THEOREM. (**CFSG**: CLASSIFICATION OF FINITE SIMPLE GROUPS.) *Each finite simple group is isomorphic to one of:*

1. *an alternating group* Alt_n*;*
2. *a classical linear group* $PSp_n(q)$, $PSU_n(q)$, $P\Omega_n^\epsilon(q)$, *or* $PSL_n(q)$*;*
3. *an exceptional group of Lie type* $E_6(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$, $G_2(q)$, ${}^2B_2(q)$, ${}^3D_4(q)$, ${}^2E_6(q)$, ${}^2F_4(q)$, *or* ${}^2G_2(q)$*;*
4. *one of* 26 *sporadic groups;*
5. *a cyclic group of prime order.*

(1.2) THEOREM. (**BBHST**: BELYAEV, BOROVIK, HARTLEY, SHUTE, THOMAS.) *Each locally finite simple group which is not finite but has a faithful representation as a linear group in finite dimension over a field is isomorphic to a Lie type group*

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$\Phi(K)$, where K is an infinite, locally finite field, that is, an infinite subfield of $\overline{\mathbf{F}}_p$, for some prime p.

(1.3) THEOREM. *Each locally finite simple group which is not linear in finite dimension but has a faithful representation as a finitary linear group over a field is isomorphic to one of:*

1. *an alternating group Alt_Ω with Ω infinite;*
2. *a finitary symplectic group $FSp_K(V,s)$;*
3. *a finitary special unitary group $FSU_K(V,u)$;*
4. *a finitary orthogonal group $F\Omega_K(V,q)$;*
5. *a special transvection group $T_K(W,V)$.*

Here K is a (possibly finite) subfield of $\overline{\mathbf{F}}_p$, for some prime p; the forms s, u, and q are nondegenerate on the infinite dimensional K-space V; and W is a subspace of the dual $V^\star$ whose annihilator in V is trivial: $0 = \{v \in V \mid vW = 0\}$.

This paper is devoted to a discussion of Theorem **??** and its proof [**?**]. The characteristic 0 case of the theorem is to be found in [**?**], the only examples being the alternating groups Alt_Ω. See Subsection **??** below and its Theorems **??** and **??**.

The second section contains a detailed discussion of the examples in the conclusion to the theorem since such intimate knowledge is needed for their reconstruction in the proof. Particular attention is paid to the root elements of each group, these being special elements of small degree from which the underlying geometry can be recovered. Typical examples of root elements are the 3-cycles of an alternating group and the transvections of a special linear group. The third section contains in turn three subsections and introduces the general tools which are most important in the proof. Its first subsection deals with Kegel covers and their properties, the most crucial being the fact that every locally finite simple group can be glued together out of finite simple groups in an appropriate manner. This observation is due to Otto Kegel. Together with the classification of finite simple groups, it accounts for the recent activity and progress in the classification theory of locally finite simple groups.

The second part of Section 3 presents a theorem which is a linear version of an old result of Jordan, who proved that a primitive permutation group of finite degree which contains an element of small support must be alternating or symmetric. An irreducible finitary group in infinite dimension can be thought of as a group generated by elements of small degree. The linear version of Jordan's theorem says that a primitive linear group in finite dimension which is generated by elements of small degree comes from a short list of groups each of which is highly geometric. The third part of Section 3 deals with ultraproducts and specifically with a theorem, proved using ultraproducts, which allows us to sew together local geometries, obtained from the subgroups of a Kegel cover, into a global geometry which will ultimately emerge as the classical geometry of the group at hand. This theorem is motivated by Mal'cev's Representation Theorem. (Since ultraproducts and the representation theorem may not be familiar to many people, we have provided an introductory appendix.)

The fourth section then contains the actual discussion of the proof of Theorem **??**. The basic outline is very simple. The general theory of Kegel covers tells us where to look for the internal subgroup geometry of the group. Our theorem of Jordan type then implies that this internal geometry is as expected. Using ultraproducts, we can build from the internal geometry an external geometry with the desired local properties. We then identify this external geometry as the expected classical geometry.

It now appears that, as hoped in [?, ?], the classification of locally finite simple groups which are finitary linear has a natural place in the general theory of locally finite simple groups. The nonfinitary groups exhibit certain "universal" behavior not seen in the finitary groups. See the present articles by Belyaev [?] and Meierfrankenfeld [?] for precise statements and discussion along these lines. The classification has also been used by Passman in his study of the semiprimitivity problem for group algebras of locally finite groups [?, ?].

Our general references for group theory and the geometry of the classical groups are the books of Aschbacher and Taylor [?, ?]. For an introduction to finitary linear groups, there is no better reference than the present article by Phillips [?].

2. Examples

We discuss in some detail the conclusions to Theorem ??, that is, the infinite dimensional finitary classical groups including the infinite alternating groups. We pay special attention to their generation by special kinds of elements of small degree — transvections, Siegel elements, and 3-cycles — which encode group theoretically much of the group's natural geometry.

2.1. Alternating Groups

The group Sym_Ω is the group of all permutations of the point set Ω. We usually write Sym_n for Sym_Ω when $\Omega = \{1, 2, \ldots, n\}$. The *support* of the permutation $g \in Sym_\Omega$ is the subset of those points in Ω which are moved by g rather than fixed. The *finitary symmetric group*, $FSym_\Omega$, is the normal subgroup of Sym_Ω containing all permutations with finite support. (Other notation exists; this is the group $Sym_0(\Omega)$ of [?].) Clearly if Ω is finite then $FSym_\Omega = Sym_\Omega$, but for infinite Ω this is false. It is the case of infinite Ω which interests us here.

The finitary symmetric group $FSym_\Omega$ is that subgroup of Sym_Ω which is generated by the class of 2-cycles and can be thought of as the union (or direct limit) of its finite subgroups Sym_Δ, as Δ ranges over the finite subsets of Ω. In particular, it is locally finite. If it were possible to find an odd number of 2-cycles whose product was the identity, this would be achieved within some subgroup Sym_Δ with Δ finite. Since this can not be the case, the members of $FSym_\Omega$ can be divided into odd and even elements, as in the finite case. The normal subgroup of Sym_Ω consisting of all even finitary permutations is the *alternating group*, Alt_Ω. It is the union of its finite alternating subgroups Alt_Δ. In particular, it is locally finite and simple.

For any field K, consider the permutation module $K\Omega$ for Alt_Ω (and $FSym_\Omega$). If the support of an element g contains t points of Ω in s distinct orbits, then the degree of g on $K\Omega$ is $t-s$. Thus the module $K\Omega$ gives rise to a finitary linear representation of Alt_Ω. (In particular, Alt_Ω is finitary in all characteristics, including 0.) On the other hand, when Ω is infinite Alt_Ω contains p-groups of unbounded class for all primes p, and so can not be linear in any characteristic. We thus have our first example of a locally finite, finitary linear simple group which is not linear in any finite dimension.

For alternating groups, the *root elements* will be the 3-cycles. These special elements of order 3 have minimal support (size 3) and minimal degree (equal to 2) in Alt_Ω. The following proposition should be thought of as saying that the geometry of the underlying set Ω is embedded group theoretically in Alt_Ω via the class of 3-cycles.

(2.1) PROPOSITION. *A subgroup of Alt_Ω which is generated by 3-cycles is a direct product of natural, alternating subgroups Alt_Σ, for $\Sigma \subseteq \Omega$.*

PROOF. Let $G \leq Alt_\Omega$ be generated by 3-cycles, and let Σ be an orbit of G on Ω. We need to show that G contains Alt_Σ.

For $\alpha, \omega \in \Sigma$, there is a set of 3-cycles $t_1, t_2, \ldots, t_n$ in G (for some n) with α in the support of t_1, ω in the support of t_n, and each pair t_i, t_{i+1} having supports with nontrivial intersection. Within $\langle t_{n-1}, t_n \rangle \leq Alt_5$, we can find a 3-cycle t^*_{n-1} whose support contains ω and meets the support of t_{n-2} nontrivially. This shortens the path from α to ω; and, proceeding in this manner, we find a 3-cycle t of G which has both α and ω in its support. If β is a third member of the orbit Σ, then similarly we find a 3-cycle s of G whose support contains β and ω. Finally in $\langle t, s \rangle \leq Alt_5$, there is a 3-cycle of G whose support is α, β, ω. Thus all 3-cycles with support from Σ belong to G, as desired. □

(2.2) COROLLARY. *(See [**?**, (4.7)], [**?**, Theorem 3].) Let $g \in Alt_n$ be an element of odd prime order p which is composed of s disjoint p-cycles, where $2sp - 1 \leq n$. Then Alt_n is generated by $2 + \lceil (n-2)/s(p-1) \rceil$ conjugates of g.*

PROOF. An easy induction shows that, for each $t \geq 2$, there are t conjugates of g in Alt_n which generate a subgroup transitive on anything up to $ts(p-1)+1 \leq n$ points. Therefore a subgroup of Alt_n with orbits of length $n-1$ and 1 can be generated by $\lceil (n-2)/s(p-1) \rceil$ conjugates of g, one of which can be g itself. One more generates with these a doubly transitive hence primitive subgroup of Alt_n. Now choose a conjugate h of g whose support intersects that of g in a set of size 1. (This is possible as $n \geq 2sp - 1$.) Then $[g, h]$ is a 3-cycle. By the proposition, the $2 + \lceil (n-2)/s(p-1) \rceil$ conjugates of g now selected must generate all of Alt_n. □

2.2. Special Transvection Groups

The alternating and finitary symmetric groups mentioned above are finitary linear over any field and are locally finite, but in general a finitary group need not be locally finite. Indeed a slight modification to an old result of Schur says that a finitary linear group is locally finite if and only if it is periodic. (See [**?**, **?**, **?**].)

Any subgroup of $FGL_K(V)$ will be locally finite if the field K is locally finite, that is, a subfield of the algebraic closure of some field of prime order, $K \leq \overline{\mathbf{F}}_p$. To see this, first note that any finite set Σ of such transformations generates a group which acts faithfully on a finite dimensional subspace of V, and the matrices for Σ in this action have only finitely many entries. These entries therefore generate a finite subfield F of locally finite K. That is, $\langle\Sigma\rangle$ is a subgroup of $GL_n(F)$, for some finite $F \le K$, and so is finite.

A transvection of $GL_K(V)$ is a nonidentity element which is as close as possible to being the identity. More precisely, a *transvection* t has $(t-1)^2 = 0$ with the range $V(t-1)$ of dimension 1. Choose a representative x of this range, $V(t-1) = \langle x\rangle$. Then $v \mapsto v(t-1) = \alpha x$ gives a linear functional $\varphi\colon v \mapsto \alpha$ such that $x\varphi = 0$ (since $(t-1)^2 = 0$).

The pair x, φ completely determines t; and, conversely, for any pair $x \in V$ and $\varphi \in V^\star$ (the dual) with $x\varphi = 0$, we have a transvection $t = t_{\varphi,x}$ given by

$$v.t_{\varphi,x} = v + (v\varphi)x\,.$$

The *K-transvection subgroup* $T(\langle\varphi\rangle, \langle x\rangle)$ is then the subgroup composed of the identity plus all transvections $t_{\varphi,\alpha x} = t_{\varphi\alpha,x}$, as α runs through the nonzero elements of the field K, and is isomorphic to the additive group of K. For $SL_K(V)$, the transvections are the *root elements* and the K-transvection subgroups are the corresponding *root subgroups*.

The transvection t is finitary of degree 1 (by definition) and is unipotent since $(t-1)^2 = 0$. In particular, if V has finite dimension then t has determinant 1. As it has degree 1, the element t can act nontrivially in at most one H-composition factor in V whenever $t \in H \le GL_K(V)$. In particular, if $H = \langle t^H\rangle$, then H has at most one nontrivial composition factor in V and so is unipotent-by-irreducible. The next lemma contains a related and important geometric property of transvections.

(2.3) LEMMA. *The transvection $t_{\varphi,x}$ leaves invariant the subspace $W \le V$ if and only if either $x \in W$ or $W \le \ker\varphi$.*

PROOF. If W is not in $\ker\varphi$, there is a $w \in W$ with $w.t_{\varphi,x} \ne w$, whence W contains

$$w.t_{\varphi,x} - w = w + w\varphi.x - w = \alpha x \ne 0\,. \quad \square$$

If $V = K^n$ is spanned by the canonical basis $e_1, \dots, e_n$ with dual basis $e_1^\star, \dots, e_n^\star$, then the transvection $t_{e_i^\star,\alpha e_j}$ is just the usual elementary matrix $I + \alpha e_{i,j}$, where $e_{i,j}$ is a matrix unit. Gaussian elimination proves:

(2.4) THEOREM. *If $\dim_K V$ is finite, then $SL_K(V)$ is generated by its transvections.*

For general V we define the *finitary special linear group* $FSL_K(V)$ to be that subgroup of $GL_K(V)$ which is generated by all transvections $t_{\varphi,x}$, with $\varphi \in V^\star$, $x \in V$, and $x\varphi = 0$. Because it is true in finite dimensions, we always have $FSL_K(V) = FGL_K(V)'$, the derived group, excepting the usual small cases. (It is possible to define a determinant function on $FGL_K(V)$ because finitary transformations have only finitely many eigenvalues not 1. The theorem then implies that the kernel of the determinant homomorphism is $FSL_K(V)$.[1])

[1] Although any unipotent element could lay claim to having determinant 1, the determinant function really only makes sense in the finitary context; so our notation is somewhat redundant. We might better use $SL_K(V)$ in place of $FSL_K(V)$, just as we use Alt_Ω over the redundant $FAlt_\Omega$. We nevertheless prefer the notation $FSL_K(V)$.

If V has finite dimension, then $V \simeq V^\star$ and $FSL_K(V) = SL_K(V)$. When V has infinite dimension then $V^\star$ has uncountably infinite dimension; and, in particular, $FSL_K(V)$ is uncountable. There is another finitary counterpart to the special linear group which remains countable for countable V, the stable special linear group $SL^0_\infty(K)$. This is best introduced in terms of matrices. Every $k \times k$ matrix A_k can be extended to a $k+1 \times k+1$ matrix A_{k+1} by placing A_k in the upper lefthand corner of A_{k+1} and then bordering A_k with 0's in A_{k+1} except for a new diagonal 1:

$$A_k \longrightarrow A_{k+1} = \left(\begin{array}{c|c} A_k & 0 \\ \hline 0 & 1 \end{array} \right) .$$

This gives us natural embeddings

$$GL_1(K) \to GL_2(K) \to GL_3(K) \to GL_4(K) \to \cdots \to GL_k(K) \to \cdots$$

The union of these groups is then the *stable linear group* $GL^0_\infty(K)$ and is countable when K is, since it is the ascending union of countable groups.

The stable linear group has a natural finitary action on the K-space V spanned by $\mathcal{B} = \{e_1, e_2, \dots, e_k, \dots\}$, where $\mathcal{B}_k = \{e_1, e_2, \dots, e_k\}$ is the standard basis for the natural module of $GL_k(K)$, for each k. Its derived subgroup (and determinant 1 subgroup) is the *stable special linear group* $SL^0_\infty(K) = GL^0_\infty(K)'$ and is the corresponding ascending union of the subgroups $SL_k(K)$.

If we think of $GL_K(V)$ as infinite matrices with respect to the basis $\mathcal{B}$, then $GL^0_\infty(K)$ is that finitary subgroup of matrices which differ from the identity only within a finite number of rows and columns. In contrast, if A is an arbitrary matrix of the finitary linear group $FGL_K(V)$, then $A - I$ will have only a finite number of nonzero columns, but there may be infinitely many rows in which A differs from the identity.

We can unify and generalize our two infinite dimensional versions of the special linear group, $FSL_K(V)$ and $SL^0_\infty(K)$, by first realizing that both are generated by K-transvection subgroups. By definition, $FSL_K(V)$ is generated by all transvections, while the theorem and our construction show that the stable group is generated by the various elementary matrix transvections $I + \alpha e_{i,j}$ (Gaussian elimination again).

Let U be a K-subspace of V and W a K-subspace of the dual $V^\star$. Then the *special K-transvection group* $T_K(W,U)$ is defined as

$$T_K(W,U) = \langle\, t_{\varphi,x} \,|\, \varphi \in W, x \in U, x\varphi = 0 \,\rangle ,$$

the subgroup of $GL_K(V)$ generated by all the transvections $t_{\varphi,x}$ where the eligible pairs φ, x are restricted to W and U. Clearly such a group is finitary. In fact it is a subgroup of $FSL_K(V) = T(V^\star, V)$. On the other hand $SL^0_\infty(K) = T(W,V)$, where W is the subspace of $V^\star$ spanned by $\mathcal{B}^\star = \{e^\star_1, e^\star_2, \dots, e^\star_k, \dots\}$, the dual of the basis $\mathcal{B} = \{e_1, e_2, \dots, e_k, \dots\}$.

There is a certain amount of symmetry here between W and U. A transvection on V is also a transvection in its natural action on $V^\star$, so in some respects it may be better to think of the transvection group G as acting on the product ${}_KU \times W_K$ respecting the natural pairing $p\colon U \times W \to K$ given by $p(u,w) = u.w$. The members of $T(W,U)$ act as isometries of $U \times W$ equipped with this pairing in the sense that, for all $u \in U$, $w \in W$, and $g \in T(W,U)$,

$$p(u,w) = u.w = ug.g^{-1}w = ug.wg = p(ug,wg)\,,$$

using the natural right action of $GL_K(V)$ on $V^\star$. The group $T(W,U)$ is faithful on both U and W if and only if the pairing p is nondegenerate on both sides, that is, if $\mathrm{ann}_U W = 0$ and $\mathrm{ann}_W U = 0$. Here by definition

$$\mathrm{ann}_U W = \{\, u \in U \mid u.w = 0, \text{ for all } w \in W \}\,,$$

and similarly for $\mathrm{ann}_W U$.

For $G = T(W,V)$ we are guaranteed $\mathrm{ann}_W V = 0$ since $W \le V^\star$. Consider the case where $\mathrm{ann}_V W$ is also 0, so that G is irreducible in its action on V by Lemma **??**. If $\dim_K V = n$ is finite, then the only possible such choice for W is the complete dual $V^\star$; and $T_K(W,V) = SL_K(V) \simeq SL_n(K)$. Assume now that $\dim_K V$ is infinite, where (as we have seen) there are various distinct choices for W with $\mathrm{ann}_V W = 0$.

Let Σ be a finite subset of G, so that $[W,\Sigma] = W_0$ and $[V,\Sigma] = V_0$ both have finite dimension. Since the pairing p is nondegenerate, there are finite dimensional W_1 and V_1, with $W_0 \le W_1 \le W$ and $V_0 \le V_1 \le V$, for which the restriction of the pairing p is nondegenerate. Thus $\langle \Sigma \rangle \le T(W_1,V_1) \le G$. Nondegeneracy then guarantees that W_1 and V_1 have the same finite dimension, k say, and that $T(W_1,V_1)$ is isomorphic to $SL_k(K)$.

Therefore $G = T(W,V)$ with $\mathrm{ann}_V W = 0$ has every finitely generated subgroup contained in a quasisimple subgroup. This forces G itself to be simple or possibly quasisimple. As G is irreducible on V, any central element is multiplication by a member of the division ring $\mathrm{Hom}_K(V,V)$. When V has infinite dimension, such a nontrivial multiplication is not finitary whereas G is. We conclude that in this case G is simple. Indeed if K is locally finite, then G is locally finite, simple, and finitary. (It can not be linear, because it has alternating sections of unbounded degree.)

In Proposition **??** we saw that the geometry of the alternating group can be reclaimed from its class of 3-cycles, its root elements. A similar statement is true here for groups generated by K-transvection subgroups, that is, root subgroups.

(2.5) THEOREM. *Let $G \le GL_K(V)$ be an irreducible group generated by the conjugacy class T^G of K-transvection subgroups with $|T| = |K| > 2$. Then G is either*

1. *$FSp_K(V,s)$, for s a nondegenerate symplectic form, or*
2. *$T_K(W,V)$, for some $W \le V^\star$ with $\mathrm{ann}_V W = 0$.*

Here $FSp_K(V,s)$ is a finitary symplectic group, as discussed in the next subsection.

This theorem is from [**?**]. (There the reducible case and the case $|T| = |K| = 2$ are also handled, but the results are more complicated. For instance, the 2-cycles of the finitary symmetric group are transvections on a natural module in characteristic 2.) For finite dimensional V, the theorem is due to McLaughlin [**?**]. Related results appear in Zalesskii and Serezhkin [**?**]. Kantor [**?**] considers the more general situation of indecomposable subgroups of orthogonal groups which are generated by Siegel elements; see the subsection on orthogonal groups below and also [**?**].

We have already observed that, for ${}_KV$ of countable dimension, the stable group $SL^0_\infty(K)$ can be realized as a group $T(W,V)$ with W of countable dimension and $\mathrm{ann}_V W = 0$. The converse holds. If W is a subspace of $V^\star$ with $\mathrm{ann}_V W = 0$ then the dimension of W must be infinite, and such a W of countable dimension has a basis which is dual to some basis of V. (See [**?**] for this and other remarks on the groups $T(W,V)$.) For W of countable dimension with $\mathrm{ann}_V W = 0$, the group $T(W,V)$ is thus isomorphic to $SL^0_\infty(K)$. Indeed, any two such subgroups are conjugate in $GL_K(V)$. Therefore although the initial construction of the stable group appears to be very basis dependent, it actually has a very natural and basis-free definition as a minimal irreducible group among the $T(W,V)$. In particular, any group $T(W,V)$ which is countable and irreducible on V must be a stable linear group $SL^0_\infty(K)$ for a suitable choice of basis $\mathcal{B}$.

2.3. Finitary Symplectic Groups

Consider the K-space V endowed with the *symplectic* (or alternating) form $s\colon V \times V \to K$. That is, s satisfies, for all $\alpha, \beta \in K$ and $x, y, z \in V$:

1. $s(\alpha x + \beta y, z) = \alpha s(x,z) + \beta s(y,z)$;
2. $s(x,y) = -s(y,x)$;
3. $s(x,x) = 0$.

In this case, the pair $({}_KV, s)$ is called a *symplectic space* (although we sometimes abuse the terminology by referring to V itself as the symplectic space). The *radical* of the symplectic space is its subspace

$$\mathrm{rad}\,({}_KV, s) = \{v \in V \mid s(v,w) = 0, \text{ for all } w \in V\}\,.$$

The symplectic space is *nondegenerate* if its radical is 0, and otherwise it is *degenerate*.

The linear transformation $g \in GL_K(V)$ is an *isometry* of the symplectic space $({}_KV, s)$ provided that, for all $x, y \in V$,

$$s(x,y) = s(xg, yg)\,.$$

The subgroup of $GL_K(V)$ consisting of all isometries is the *symplectic group* $Sp_K(V,s)$. The *finitary symplectic group* is then the group of isometries which are finitary:

$$FSp_K(V,s) = Sp_K(V,s) \cap FGL_K(V)\,.$$

For a given V of uncountable dimension, there are many fundamentally different nondegenerate symplectic forms and so different groups. The countable dimension case mimics that of finite dimension, in that there is a unique nondegenerate form up to similarity, and so a unique nondegenerate symplectic group up to isomorphism. (See [**?**].)

The null axiom $s(x,x) = 0$ for symplectic spaces is an immediate consequence of the preceding alternating axiom $s(x,y) = -s(y,x)$ when K has characteristic other than 2. Symplectic spaces in characteristic 2 have other exotic properties. In particular, the symplectic groups in characteristic 2 also arise as orthogonal groups (see below); and it will on occasion be convenient to restrict consideration of the symplectic case to characteristics other than 2, leaving the treatment of characteristic 2 to the orthogonal case.

Let $({}_KV, s)$ be a nondegenerate symplectic space. For each vector $x \in V$, the map $\delta_x : V \to K$ given by mapping v to $s(v, x)$ is a K-linear functional, an element of $V^\star$. The map $\delta : x \mapsto \delta_x$ is then a canonical K-isomorphism of V into $V^\star$. If the transvection $t_{\varphi,x}$ is an isometry of (V, s), then it is in an easy exercise to see that

$$x^\perp = \{v \in V \mid s(v, x) = 0\}\,,$$

the kernel of the functional δ_x, must also be the kernel of the functional φ. That is, $t_{\varphi,x}$ is a symplectic isometry if and only if $\varphi = \delta_x\alpha$, for some $\alpha \in K$. The symplectic *root elements* are then the *symplectic transvections* $t_{\delta_x\alpha,x} = t_{\delta_x,\alpha x}$ which we write as

$$t_{x,\alpha x} : v \to v + s(v, x)\alpha x\,.$$

The corresponding *root subgroup* T_x is the K-transvection subgroup consisting of the identity and the elements $t_{x,\alpha x}$, as α runs through the nonzero elements of K. Again it is isomorphic to the additive group of K.

(2.6) THEOREM. *(See [**?**, 8.5,8.8].) For a finite dimensional and nondegenerate symplectic space $({}_KV, s)$, the symplectic group $Sp_K(V, s)$ is generated by its root elements, the symplectic transvections. The symplectic group is quasisimple, except for certain small, finite exceptions, and has center $\{\pm 1\}$.*

From this we easily conclude:

(2.7) COROLLARY. *For a nondegenerate and infinite dimensional symplectic space $({}_KV, s)$, the finitary symplectic group $FSp_K(V, s)$ is simple and generated by its root elements, the symplectic transvections.*

We have already anticipated this corollary in Theorem **??** where finitary symplectic groups and the groups $T(W, V)$ were characterized as those irreducible groups generated by K-transvection subgroups. Unlike the special linear case, for a given nondegenerate symplectic space $({}_KV, s)$ we do not have several different finitary analogues, but only the one. This is because the "duality" δ identifies a canonical subspace $W = \delta(V)$ of the dual $V^\star$ for which $FSp(V, s) \le T(W, V)$. In particular, for V of countable dimension the finitary symplectic group is isomorphic to the stable symplectic group, which can be constructed by embedding $Sp_{2k}(K)$ in the upper lefthand corner of $Sp_{2k+2}(K)$ as before. (Extend the nondegenerate symplectic space $V_{2k} = K^{2k}$ to $V_{2k+2} = K^{2k+2}$ by adding on a perpendicular direct summand which is nondegenerate of dimension 2.)

Our next result is a symplectic relative of Corollary **??**. It states that, in a strong sense, the minimum possible number of root element (transvection) generators is in general enough.

(2.8) THEOREM. *Let $G = Sp(V, s) \simeq Sp_{2n}(q)$, where q is odd but $(n, q) \neq (1, 9)$. For t a transvection, consider a subgroup $H = \langle t^H \rangle$ with $\dim[V, H] = k < 2n$. Then there is a set $\Sigma = \{t_1, \ldots, t_{2n-k}\}$ of $2n - k$ distinct G-conjugates of t, such that $G = \langle H, \Sigma \rangle$.*

In the exceptional cases where q is even or $(n, q) = (1, 9)$, there is a suitable Σ of size $2n - k + 1$.

Proof. The case $n = 1$ only asks how many transvections are required to generate $SL_2(q)$. The answer [**?**, Theorem 4.9] is two, if q is odd but not 9, and three otherwise.

In the nonexceptional general case (q odd but not 9), we observe that from the case $n = 1$ there is a t_1 with $H_1 = \langle H, t_1 \rangle$ satisfying $\dim[V, H_1] = k+1$ and and such that H_1 contains a subgroup $Sp_2(q)$ and hence a full K-transvection subgroup. Now the result follows easily from Theorem **??** and indeed from MacLaughlin's original result [**?**].

For $Sp_4(9)$, the result should be checked by hand, and then an argument as in the preceding paragraph takes over for $(n, q) = (\geq 3, 9)$. Similarly, for q even, the previous argument works as well, provided we are willing to accept one additional generator. □

For instance, taking $H = \langle t \rangle$, we learn that G can always be generated by $2n$ distinct transvections except for $(n, q) = (1, 9)$ and q even where $2n + 1$ suffice. Any number of transvections smaller than $2n$ would generate a subgroup whose commutator was proper in V and so could not be all of G. Although we have not proven it, in characteristic 2 the additional transvection generator is always needed. This is associated with the fact that symplectic groups in characteristic 2 are also orthogonal groups, as will be discussed further below. In $Sp_2(9) \simeq Alt_6$ transvections are 3-cycles, and three are required for generation.

For the finite classical groups of other types, similar theorems hold. Theorem **??** and Proposition **??** include special cases.

2.4. Finitary Unitary Groups

Let σ be an automorphism of order 2 of the field K. We provide the K-space V with a *unitary* (or hermitian) form $u: V \times V \to K$. That is, u satisfies, for all $\alpha, \beta \in K$ and $x, y, z \in V$:

1. $u(\alpha x + \beta y, z) = \alpha u(x, z) + \beta u(y, z)$;
2. $u(x, y) = u(y, x)^\sigma$.

In this case, the pair $({}_K V, u)$ is called a *unitary space* (briefly, V is a unitary space). The *radical* of the unitary space is, as before,

$$\mathrm{rad}\,({}_K V, u) = \{v \in V \mid u(v, w) = 0, \text{ for all } w \in V\}\,.$$

The unitary space is *nondegenerate* if its radical is 0, and otherwise it is *degenerate*.

The linear transformation $g \in GL_K(V)$ is an *isometry* of the unitary space $({}_K V, u)$ provided that, for all $x, y \in V$,

$$u(x, y) = u(xg, yg)\,.$$

The subgroup of $GL_K(V)$ consisting of all isometries is the *general unitary group* $GU_K(V, u)$. The *finitary unitary group* is then the group of isometries which are finitary:

$$FGU_K(V, u) = GU_K(V, u) \cap FGL_K(V)\,.$$

The notion of unitary space which we study is not the most general. It is not necessary to restrict attention to fields rather than division rings. More seriously, we shall only consider those unitary spaces which contain isotropic vectors. A nonzero vector x is called *isotropic* if $u(x, x) = 0$. A complex space endowed with its usual inner product provides an example of a unitary space which has no isotropic vectors. Such a space is called *anisotropic*. The isometry groups of anisotropic spaces behave in general differently from those of our unitary spaces. We are interested primarily in groups and spaces over finite or locally finite fields, and over such a field any anisotropic space must have dimension 1.

Let $({}_KV, u)$ be a nondegenerate unitary space. For $\alpha \in K$ and $x \in V$, the transvection

$$t_{x,\alpha x}: v \to v + u(v, x)\alpha x$$

is an isometry if and only if x is isotropic and $\alpha^\sigma = -\alpha$ (α is called *skew*). These *unitary transvections* are our *root elements* in this case. For a fixed isotropic 1-space $\langle x \rangle$, the corresponding *root subgroup* is the subgroup consisting of the identity and the elements $t_{x,\alpha x}$, as α runs through the nonzero skew elements of K. The root subgroup is isomorphic to the additive group of $K_0 = \{\beta \mid \beta = \beta^\sigma\}$, the degree 2 subfield of K composed of elements fixed by the automorphism σ.

In finite dimensions, a unitary isometry need not have determinant 1, so we are also interested in the *special unitary* group

$$SU_K(V, u) = GU_K(V, u) \cap SL_K(V)$$

or, more generally, in the *finitary special unitary group*

$$FSU_K(V, u) = GU_K(V, u) \cap FSL_K(V)\,.$$

(2.9) THEOREM. *(See [**?**, 10.23].) Let $({}_KV, u)$ be a finite dimensional and nondegenerate unitary space which contains an isotropic vector. Then, except for certain small, finite exceptions, the special unitary group $SU_K(V, u)$ is quasisimple and is generated by its root elements, the unitary transvections.*

(2.10) COROLLARY. *If the nondegenerate and infinite dimensional unitary space $({}_KV, u)$ contains an isotropic vector, the finitary special unitary group $FSU_K(V, u)$ is simple and is generated by its root elements, the unitary transvections.*

As before, subgroups of unitary groups which are generated by transvections must have a very restricted structure. For instance, the minimum conceivable number of transvection generators for the full special unitary group can be achieved in most cases.

(2.11) PROPOSITION. *(See [**?**, Theorem 4.9].) If $\dim_K V \geq 5$, K is finite, and u is nondegenerate, then $SU_K(V, u)$ is generated by n unitary transvections.*

A more general result like Theorem **??** is true here as well.

For the finite group $SU_K(U, u)$ of the proposition, the field K must be $\mathbf{F}_{r^2}$, for some prime power r; and the form u is uniquely determined up to isometry. The group is thus unique up to isomorphism and is often denoted by $SU_n(r^2)$ or by $SU_n(r)$. For the purposes of this paper we prefer and use exclusively the first of these two differing pieces of notation. In particular the natural module for $SU_n(q)$ is defined over $\mathbf{F}_q$.

2.5. Finitary Orthogonal Groups

On the K-space V the *orthogonal space* (or, sometimes, quadratic space) $({}_KV, q)$ with *quadratic* form $q: V \to K$ and associated *orthogonal* (or symmetric bilinear) form $b: V \times V \to K$ satisfies, for all $\alpha, \beta \in K$ and $x, y, z \in V$:

1. $b(\alpha x + \beta y, z) = \alpha b(x, z) + \beta b(y, z)$;
2. $b(x, y) = b(y, x)$;
3. $q(\alpha x) = \alpha^2 q(x)$;
4. $q(x + y) - q(x) - q(y) = b(x, y)$.

Clearly q uniquely determines b, the *polar form* of q. Conversely we have

$$2q(x) = q(2x) - q(x) - q(x) = b(x, x)\,.$$

Therefore if the characteristic of K is not 2, the orthogonal form b uniquely determines q. In characteristic 2 the polar form of q is revealed as symplectic since $2q(x) = 0$. Conversely, starting with any nonzero symplectic b and a map q defined on a basis, we can extend q to a quadratic form with b as it polar form by using 3. and 4. above. In particular, in characteristic 2 any symplectic b will be the polar form for more than one quadratic form q.

The *radical* of the orthogonal space $({}_KV, q)$ with polar form b is

$$\mathrm{rad}\,({}_KV, q) = \{v \in V \mid q(v) = 0,\ b(v, w) = 0,\ \text{for all } w \in V\}\,.$$

The orthogonal space is *nondegenerate* if its radical is 0, and otherwise it is *degenerate*. In characteristic not 2, the radical of the quadratic form q equals the radical of its polar form b (with the obvious definition). In characteristic 2, the form b is symplectic on V and $\mathrm{rad}\,({}_KV, q)$ could be strictly smaller than the symplectic radical $\mathrm{rad}\,({}_KV, b)$. If the symplectic radical is 0, then the space $({}_KV, q)$ is *nondefective*; otherwise it is *defective*.

The axioms imply that the restriction of a nondegenerate but defective quadratic form q to the symplectic radical is a σ-semilinear map whose image is a K^2-subspace of K, where K^2 is the subfield of squares in K. In the cases of interest to us, K^2 will always be equal to K; and the symplectic radical will have K-dimension 1 (or 0).

The linear transformation $g \in GL_K(V)$ is an *isometry* of the orthogonal space $({}_KV, q)$ provided that, for all $x \in V$,

$$q(x) = q(xg)\,.$$

The subgroup of $GL_K(V)$ consisting of all isometries is the *general orthogonal group* $GO_K(V, q)$. The *finitary general orthogonal group* is then the group of isometries which are finitary:

$$FGO_K(V, q) = GO_K(V, q) \cap FGL_K(V)\,.$$

We may again pass to the special orthogonal group of determinant 1 isometries, but here this subgroup need not be perfect. We are more interested in the derived group

$$F\Omega_K(V, q) = FGO_K(V, q)'$$

which we shall call the *finitary orthogonal group.* (For V of finite dimension we write $\Omega_K(V,q) = GO_K(V,q)'$. In certain small, finite cases, this notation may go against convention.)

In characteristic other than 2, a nondegenerate orthogonal space has no isometries which are transvections; in characteristic 2, a K-transvection subgroup meets the general orthogonal group in a subgroup of order at most 2. Let $({}_KV,q)$ be a nondegenerate orthogonal space. The subspace U of V is *totally singular* if the restriction of q to U is identically 0, so that U is its own radical. In an orthogonal group, a *root element* s is a unipotent element with $(s-1)^2=0$ for which the range $V(s-1)$ is totally singular of dimension 2. We shall refer to these orthogonal root elements as *Siegel elements.* (In [**?**, Chap. 11] these are the Siegel transformations of type II.) The *root subgroups* correspond to the various totally singular 2-spaces and are isomorphic to the additive group of K. It is possible to give a precise formula for the action of a Siegel element just as we have done earlier for transvections, but we shall not require the exact description.

Let $({}_KV,q)$ be a nondefective orthogonal space in characteristic 2 with polar symplectic form b. Choose $x \in V$ with $q(x) \neq 0$. Let W be the subspace $x^\perp = \{v \in V \mid b(v,x) = 0\}$ and p the restriction of q to W. The orthogonal space $({}_KW,p)$ is then nondegenerate but defective with symplectic radical $\langle x\rangle$. The space $\bar{W} = W/\langle x\rangle$ is a nondegenerate symplectic space with form $\bar{b}$ induced by the restriction of b to W. Any symplectic isometry of the space $\bar{W}$ then lifts uniquely to an orthogonal isometry of W. Therefore the nondegenerate symplectic space $({}_K\bar{W},\bar{b})$ and its group can be realized as a nondegenerate orthogonal space $({}_KW,p)$ and its group. In fact any symplectic group in characteristic 2 can be "made orthogonal" by this process. As mentioned above, we often prefer to think of symplectic groups in characteristic 2 as orthogonal groups. There is the possibility of confusion here because there are two different types of root elements involved — the symplectic transvections and the orthogonal Siegel elements. Each transvection on symplectic $({}_K\bar{W},\bar{b})$ lifts to a transvection on orthogonal $({}_KW,p)$, but two different transvections of the symplectic root subgroup $T_{\bar{z}}$ necessarily lift to t_{φ_1,z_1} and t_{φ_2,z_2} with $\langle z_1\rangle \neq \langle z_2\rangle$.

(2.12) THEOREM. *(See [**?**, 8.8,11.9,11.48].) Let $({}_KV,q)$ be a finite dimensional and nondegenerate orthogonal space which has dimension at least 5 and contains totally singular 2-spaces. In characteristic 2, assume further that the dimension is at least 6 and that $K=K^2$. The orthogonal group $\Omega_K(V,q)$ is quasisimple and, in particular, is generated by its root elements, the Siegel elements.*

(2.13) COROLLARY. *If $({}_KV,q)$ is a nondegenerate and infinite dimensional orthogonal space which contains a totally singular 2-space and also in characteristic 2 has $K=K^2$, then the finitary orthogonal group $F\Omega_K(V,q)$ is simple and generated by its root elements, the Siegel elements.*

For the finite and nondegenerate orthogonal groups, there are results about subgroups generated by root elements similar to Theorems **??** and **??** and Proposition **??** above. In particular, the minimum conceivable number of root element generators (namely, one-half the dimension) is always nearly enough. See also [**?**, **?**].

3. Tools

In this section we discuss the main tools of our proof — Kegel covers; a linear theorem of Jordan type; and representations constructed via ultraproducts.

3.1. Kegel Covers

Let G be a simple group and let X be a countable (or finite) subset of G. It will be no disadvantage to assume that X is a subgroup, since every countable subset of a group generates a countable subgroup. The simplicity of G is in fact a local property, in the sense that it can be checked within the finitely generated subgroups of G:

> *The group G is simple if and only if, for each ordered pair $x, y \in G$, there is a finite set $g_1, \ldots, g_k$ (for some k which depends upon x and y) with $x = \prod_{i=1}^{k} y^{\pm g_i}$.*

With this in mind, we try to build $X = X_0$ into a simple group. For each pair x, y from X_0, find $g_1, \ldots g_k$ as described; and let X_1 be the subgroup generated by X_0 together with all the g_i required for the various pairs x, y. As X_0 is countable, the number of pairs is countable, as is the total number of the various g_i; and so, ultimately, the subgroup X_1 is itself countable. We have designed X_1 to verify simplicity relative to all pairs x, y from its subgroup X_0, but in the process many more elements have presumably been introduced. We continue in the same manner, but now starting from the subgroup X_1. Choose in turn each of the countable number of pairs $x, y \in X_1$; find suitable g_i for each pair; and define the new subgroup X_2 to be generated by X_1 together with all the new g_i. Now we have a countable X_2 which verifies simplicity relative to each pair x, y from its subgroup X_1. This is a machine whose crank we can keep on turning, at each new stage i producing a countable subgroup X_i which verifies simplicity relative to all pairs from its subgroup X_{i-1}. If we now set $H = \bigcup_i X_i$, then the subgroup H of G verifies simplicity relative to each pair of its elements; and so H itself is simple. It contains the original subgroup X and, being the ascending union of countable subgroups, is countable itself. We have proven:

(3.1) Theorem. (P. Hall) *In a simple group, every countable subset is contained in a countable simple subgroup.*

This argument, which is due to Phillip Hall, is in fact a special case of the downward Lowenheim-Skolem argument of model theory. There are several places in the theory of locally finite simple groups where ideas and methods from model theory appear to great advantage. Another important instance is found in the third subsection (and the appendix).

Hall's result is for arbitrary simple groups, whereas we are mainly concerned with locally finite groups which are simple. We thus ask whether something stronger is true in this smaller class. We can hope that, in a locally finite simple group, every finite subgroup is contained in a finite simple subgroup. This is false; see Corollary **??** for the counterexample discovered by Zalesskii and Serezhkin. A slightly weaker statement is true, as was first proven by Kegel [**?**, 4.3]. The subgroup X is said to be *covered* by the section H/M if X is a subgroup of H which meets the normal subgroup M trivially. Such a covering leads to an isomorphic embedding of X in the section H/M.

(3.2) PROPOSITION. *In a locally finite simple group, every finite subgroup is covered by a simple section of a finite subgroup.*

PROOF. The proof of Hall's result can be attempted. At each stage the subgroup X_i is finitely generated and so finite, but the proof falls down at the last moment. The ascending union of finite subgroups still will likely be countably infinite rather than finite.

The answer is to stop along the way. First, consider the embedding of X in X_1. Since every nontrivial element of X is in the X_1-normal closure of each other element of X, there is a X_1-chief factor P/Q which covers X. It is tempting to choose $H = P$ and M a maximal normal subgroup of P containing Q, but this may not be good enough. The section P/Q is a direct product of isomorphic finite simple groups, but certain elements of X might project trivially onto the direct factor S selected by our choice of M. That is, if the element x of X projects nontrivially onto S, it is still possible for $y \in X$ to project trivially onto S and have $x \in \langle y^{g_1}, \dots, y^{g_k} \rangle$ since the g_i, while all belonging to X_1, are not necessarily in its subgroup P.

To remedy the problem, we go one step further, considering the embedding of X_1 (and its subgroup X) in X_2. Again there is a chief factor $\tilde{P}/\tilde{Q}$ of X_2 which covers X_1 and is a direct product of isomorphic simple groups. Consider again our pair x, y from the original X, and let $\tilde{S}$ now be a simple direct factor of $\tilde{P}/\tilde{Q}$ onto which x projects nontrivially. Since all the g_i from the previous paragraph are in X_1, they are in $\tilde{P}$ and are covered by the section. If y projected trivially onto $\tilde{S}$, then x could not be in its $\tilde{P}$-normal closure, which is the case since $X_1 \leq \tilde{P}$. We conclude that y must also project nontrivially onto $\tilde{S}$. That is, once one element of X projects nontrivially onto $\tilde{S}$, they all must. If we set $H = \tilde{P}$ and take M to be that maximal normal subgroup of H containing $\tilde{Q}$ which picks up all the factors of $\tilde{P}/\tilde{Q}$ except $\tilde{S}$, then X is covered by the simple section $H/M \simeq \tilde{S}$ of finite $\tilde{P}$, as required. □

A *sectional cover* of a group G is a set $\mathcal{C} = \{(G_i, N_i) \mid i \in I\}$ of pairs of subgroups such that, for each $i \in I$, N_i is normal in G_i, and, for every finitely generated subgroup X of G, there is an i with X covered by the section G_i/N_i. (If each N_i equals 1, then we speak of a *subgroup cover*.) The previous proposition then says that the locally finite simple group G has a sectional cover by simple sections G_i/N_i of finite subgroups G_i. Such a sectional cover is called a *Kegel cover* after Otto Kegel, who first proved their existence. (See [**?**, 4.3] and also [**?**, **?**].) The subgroups N_i are the *Kegel kernels*, and the simple factors G_i/N_i are the *Kegel quotients*. If G is finite, then any Kegel cover contains the trivial cover $\{(G, 1)\}$.

The Kegel cover $\mathcal{C}$ of G is a particular type of *local system* for G, which is to say, G is the union of its subgroups G_i, and for any two G_i and G_j there is a third G_k with $\langle G_i, G_j \rangle \leq G_k$. A Kegel cover has a stronger property: we can choose k so that G_k not only contains the subgroup $\langle G_i, G_j \rangle$ but the simple section G_k/N_k covers $\langle G_i, G_j \rangle$. Once such containment and covering statements are true about pairs of subgroups G_i, G_j, they actually hold for all finite collections of subgroups G_i; there is a k such that the section G_k/N_k covers the subgroup $\langle G_1, G_2, \dots, G_n \rangle$.

The group G is the direct limit of the members of any local system. Indeed the study of covers and local systems really stems from a desire to refine the trivial observation that any group is the direct limit of its set of finitely generated subgroups. The next result is actually a combinatorial result about partial orderings in which finite sets have upper bounds, but it is used frequently and crucially to replace a large Kegel cover by a smaller and more manageable one.

(3.3) LEMMA. (COLORING ARGUMENT) *Let G be a locally finite simple group, and suppose that the pairs of the Kegel cover $\mathcal{K} = \{(G_i, N_i) \mid i \in I\}$ are colored with a finite set $1, \ldots, n$ of colors. Then $\mathcal{K}$ contains a monochromatic subcover. That is, if $\mathcal{K}_j$ is the set of pairs from $\mathcal{K}$ with color j, for $1 \leq j \leq n$, then there is a color j for which $\mathcal{K}_j$ is itself a Kegel cover of G.*

PROOF. Otherwise, for each j, there is a finite subgroup X_j of G which is not covered by any section which is colored by j. The subgroup $X = \langle X_1, \ldots, X_j, \ldots, X_n \rangle$ is therefore not covered by a section with any of the colors $1, 2, \ldots, n$. As X is generated by a finite number of finite groups, it is finite itself. Therefore some section of the Kegel cover $\mathcal{K}$ covers X, a contradiction which proves the lemma. $\square$

A trivial example of the coloring argument is nevertheless instructive. Choose a nonidentity g in G and color the sections (that is, pairs) of the Kegel cover $\{(G_i, N_i) \mid i \in I\}$ with two colors, one indicating that $g \notin G_i - N_i$ and the second color indicating $g \in G_i - N_i$. One of the color classes must be a subcover, but by definition the first can not, since it nowhere covers $\langle g \rangle$. We have proven that the sections which cover a specific element are themselves a Kegel cover. (This would have worked with any finite X in place of g. Less evident applications of the argument appear later.)

(3.4) PROPOSITION. *Let G be a locally finite simple group with Kegel cover $\mathcal{K} = \{(G_i, N_i) \mid i \in I\}$, and choose a nonidentity g in G. For each $j \in J = \{j \in I \mid g \in G_j - N_j\}$, set $H_j = \langle g^{G_j} \rangle$ and $M_j = H_j \cap N_j$. Then $\mathcal{K}_g = \{(H_j, M_j) \mid j \in J\}$ is a Kegel cover whose collection of Kegel quotients is contained in that of the original cover.*

PROOF. As G is the direct limit of the G_j, the normal closures H_j are also directed. If H is the subgroup of G which is the direct limit of the H_j, then H must be normal in G and nontrivial since $g \in H$. Therefore $H = G$. Certainly

$$H_j/M_j = H_j/H_j \cap N_j \simeq H_j N_j/N_j = G_j/N_j \,.$$

It remains to observe that if G_i/N_i covers G_j, then H_i/M_i covers H_j. $\square$

Proposition **??** tells us that Kegel covers exist. Proposition **??** and the coloring argument of Lemma **??** then allow us to begin searching for covers which are in some sense nice. The next observation [**?**, 2.8] will give us further opportunity to specialize our covers.

(3.5) LEMMA. *Let N be normal in the finite group G with G/N perfect. Then G has a unique subgroup H which is minimal subject to being normal in G and a supplement to N.*

PROOF. Let $\bar{G} = G/N$. If H_1 and H_2 are normal supplements to N, then

$$\overline{H_1 \cap H_2} \geq \overline{[H_1, H_2]} = [\bar{H}_1, \bar{H}_2] = [\bar{G}, \bar{G}] = \bar{G}\,. \ \square$$

In this case, we call $H = H(G,N)$ the *heart* of the pair (G,N). If

$$\mathcal{S} = \{(G_i, N_i) \,|\, i \in I\}$$

is a Kegel cover of the group G, then the *heart* of $\mathcal{S}$ is

$$\mathcal{H} = \{(H(G_i,N_i), N_i \cap H(G_i,N_i)) \,|\, i \in I\}\,.$$

The quotients of $\mathcal{H}$ are identical to those of $\mathcal{S}$.

For $G = Alt_\Omega$, let the *canonical cover* of G be

$$\mathcal{CC}(Alt_\Omega) = \{(Alt_\Delta, 1) \,|\, \Delta \subseteq \Omega,\ |\Delta| < \infty\}\,.$$

Notice that this is actually a cover by simple subgroups, not just simple sections.

(3.6) PROPOSITION. *Let $\mathcal{S}$ be a Kegel cover of $G = Alt_\Omega$, for some infinite Ω; and let $\mathcal{H}$ be the heart of $\mathcal{S}$. Then $\mathcal{H} \cap \mathcal{CC}(G)$ is a Kegel cover of G, and $\mathcal{H} - \mathcal{CC}(G)$ is not a cover.*

PROOF. Let t be a 3-cycle of G, and let $\mathcal{S}_t = \{(H_i, M_i) \,|\, i \in I\}$ be the Kegel cover produced as in Proposition **??** by taking the normal closure of t for each section G_i/N_i of $\mathcal{S}$ which covers t. Each $H_i = \langle t^{G_i} \rangle$ is a normal supplement to N_i in G_i, so H_i contains the heart of (G_i, N_i).

By Proposition **??** the group H_i is a direct product of natural finite alternating subgroups of G. As M_i is normal in H_i with H_i/M_i simple, M_i can only be a direct product of all but one of the direct factors. Let Alt_{Δ_i} be the missing factor. By construction H_i is normal in G_i, and so $M_i = H_i \cap N_i$ is also. This forces the lone missing factor Alt_{Δ_i} to be normal itself. The 3-cycle t is in H_i but not in N_i, so Alt_{Δ_i} must be the normal closure of t in G_i and is clearly a simple complement to N_i in G_i. That is, $H_i = Alt_{\Delta_i}$ is the heart of (G_i, N_i); $M_i = 1$; and the cover $\mathcal{S}_t$ is contained in $\mathcal{H} \cap \mathcal{CC}(G)$. On the other hand, $\mathcal{H} - \mathcal{CC}(G)$ does not cover t and so can not be a cover. $\square$

The proposition is a recast version of [**?**, (8.1)] with the elementary proof promised there. Among other things, it provides an elementary proof of Hartley's observation [**?**] that the infinite alternating groups can not be written as a nonnatural limit of diagonally embedded finite alternating groups.

Each of the simple groups G discussed in Section 2 has a Kegel cover, $\mathcal{CC}(G)$, which is a canonical cover in that every cover can be viewed as a modified version of this particular one. We present only those for the symplectic groups.

Let s be a nondegenerate symplectic form on the vector space V, infinite dimensional over the locally finite field K; and let $G = FSp_K(V, s)$ be the associated finitary symplectic group. If $\{(G_i, N_i) \mid i \in I\}$ is a Kegel cover of G, then G is the direct limit of its finite subgroups G_i and V is the direct limit of its finite dimensional subspaces $[V, G_i]$. As s is nondegenerate, every finite dimensional subspace is contained in one for which the restriction of s is nondegenerate. This implies that G has a cover by finite dimensional quasisimple symplectic subgroups. For the alternating groups, the canonical cover was composed of simple groups; but here nothing like that is true. Although as just seen quasisimple covers exist, not every cover reduces to a quasisimple one. This is essentially because the space V, although nondegenerate, can nevertheless be written as a direct limit of finite dimensional subspaces, each of which is degenerate. The associated subgroup cover is then not quasisimple but instead looks to be composed of arbitrary finite dimensional "parabolic subgroups," groups which are unipotent-by-quasisimple. Such subgroups do provide a canonical cover.

Choose a finite subset $X = \{u_1, \ldots, u_n\}$ of V, and set

$$K_X = \langle s(u_i, u_j) \mid 1 \le i, j \le n \rangle ,$$

that subfield of K generated by the entries in the Gram matrix of the set X. The subfield K_X is finite, since X is finite and K is locally finite. Next choose a finite subfield F of K which contains K_X. Set $FX = \sum_{i=1}^n Fu_i$, an F-subspace of V. We then let $G_{X,F} = \langle t_{u,\alpha u} \mid u \in FX - \mathrm{rad}\, FX,\ \alpha \in F - 0\rangle$.

The group $G_{X,F}$ can be viewed as being generated by full transvection subgroups over F, and so is a unipotent normal subgroup P extended by a symplectic group over F (see Lemma **??** and Theorem **??**). In particular, $G_{X,F}$ is unipotent-by-quasisimple. Its solvable radical $N_{X,F}$ is the unipotent radical P extended by the subgroup of symplectic scalars $\{\pm 1\}$ (which is 1 in characteristic 2). We then have the canonical cover

$$\mathcal{CC}(G) = \{(G_{X,F}, N_{X,F}) \mid X \subset V,\ |X| < \infty,\ K_X \le F \le K,\ |F| < \infty\} ,$$

and the counterpart of the previous result is valid.

(3.7) PROPOSITION. *Let $\mathcal{S}$ be a Kegel cover of $G = FSp_K(V, s)$, and let $\mathcal{H}$ be the heart of $\mathcal{S}$. Then $\mathcal{H} \cap \mathcal{CC}(G)$ is a Kegel cover of G, and $\mathcal{H} - \mathcal{CC}(G)$ is not a cover.*

As a corollary we have the result of Zalesskii and Serezhkin [**?**] mentioned earlier.

(3.8) COROLLARY. *A stable symplectic group over a finite field of odd order does not have a cover by simple subgroups.*

More generally, a finitary symplectic group in infinite dimension over a locally finite field of odd characteristic does not have a cover by simple subgroups. Indeed in odd characteristic the group $G_{X,F}$ is never simple; even when its unipotent radical is trivial, it has a central subgroup of order 2.

Belyaev [**?**] and Meierfrankenfeld [**?**] have proven by elementary methods that every locally finite simple group which is finitary has a Kegel cover whose members are unipotent-by-quasisimple, as is the case with $\mathcal{CC}(G)$. It is easy to see that the subset $\mathcal{CC}_{qs}(G)$ composed of those $G_{X,F}$ which are quasisimple (corresponding to FX which are nondegenerate for the restriction of s) is still a cover. Indeed from Theorem **??** we may conclude that every locally finite, finitary simple group has a quasisimple cover; but we do not have an elementary proof of this.

As mentioned before, for each classical finitary group there is a similarly defined canonical cover and a corresponding proposition stating that every Kegel cover is at heart canonical. This responds in a precise manner to Hartley's remark [**?**] that, for finitary locally finite simple groups, Kegel covers should be essentially unique. All the proofs are basically the same. Once a member of the cover has root elements outside its kernel, its heart must be of a very restricted form.

3.2. A Theorem of Jordan Type

A group which has a faithful representation in finite dimension will in general have many fundamentally different ones coming from, for instance, tensor, symmetric, and exterior powers. This is not the case for finitary groups which are not finite dimensional. Each simple group in the conclusion of Theorem **??** has only one basic finitary representation. All others come from this one via direct sums, "duality," or playing with the field [**?**, **?**, **?**].

We can think of an infinite dimensional, finitary group as being generated by "elements of small degree." Jordan [**?**, Théorème II] proved that a finite, primitive subgroup of Sym_n which contains an element of small support is either Alt_n or Sym_n. This idea was extended by Wielandt [**?**] who proved that a primitive subgroup of Sym_Ω, for infinite Ω, which contains a finitary permutation must in fact contain all of Alt_Ω. In our classification we need a theorem from [**?**] providing a result similar to Jordan's for finite, primitive linear groups. This theorem is really all of the classification of finite simple groups, **CFSG**, that is required for the proof of Theorem **??**. (More is of course needed for Theorem **??**, **BBHST**; and most conceivable applications of Theorem **??** would also require **BBHST**.)

(3.9) Theorem. *Let perfect finite $H \le GL_K(V)$, for K algebraically closed, with H primitive on V. Assume that H is generated by elements whose degree on V is less than $\sqrt{n}/12$, where $n = \dim_K V$. Then either:*

1. *H is Alt_n and V is a natural module; or*
2. *H is a quasisimple classical group in the same characteristic as K, and V is a nearly natural module.*

If finite G is a quasisimple classical group $Cl_n(q)$ or simple $PCl_n(q)$, then a *natural* module for G is the module $\mathbf{F}_q^n$ for its defining projective representation (or any twist of this module via automorphisms). A *nearly natural* module is a natural module tensored up to a (possibly) larger field. If G is an alternating or symmetric group on Ω, then a *natural* module for G is the nontrivial irreducible factor in the permutation module $K\Omega$, for any field K.

The irreducible, imprimitive case of the theorem can be handled as well.

(3.10) Proposition. *Let H, V, and K be as in Theorem **??**, except that the representation of H on V is assumed to be irreducible but not primitive. Let $\Omega =$*

$\{V_i \mid i \in I\}$ be a maximal block system. Then in its action on Ω the group H induces Alt_Ω, and so H is the extension of a subdirect product of $|I|$ copies of $L \leq GL_K(V_i)$ by Alt_Ω.

PROOF. The permutation action of H on Ω must involve a characteristic 0 solution to the primitive case. Therefore by Theorem **??** (or Jordan's theorem), H induces Alt_Ω on Ω. $\square$

A representation and module as in the proposition will be called *generalized monomial*. The representation and module are *monomial* if all the V_i have dimension 1, so that L is cyclic. Any permutation module $K\Omega$ is monomial for Alt_Ω, and on occasion we shall abuse notation mildly by including the natural module (and any other nontrivial section of the permutation module) as a monomial module.

It has been observed by Phillips [**?**, (9.1)] that a stronger version of the proposition (with essentially the same proof) is valid in the broader context of finitary groups which are irreducible but imprimitive and are generated by elements of small degree. There finiteness plays no role and local finiteness is replaced by the weaker assumption that there are no noncyclic free subgroups. For a careful discussion of imprimitive modules, see Phillips' article. They arise naturally in the study of general finitary groups; for instance, an irreducible, locally solvable finitary group is either finite dimensional or imprimitive in countable dimension [**?**, (8.2.5)].

3.3. Ultraproducts

In the previous two subsections we have seen that each group G from Theorem **??** is highly geometric. The linear algebra and geometry of G is essentially unique, as seen in the previous subsection, and precisely dictates the internal subgroup structure of G, as seen in the first subsection. In classifying these groups we seek to reverse the process by rebuilding the geometry out of the subgroup structure. We do this using the ultraproduct construction for groups and representations. From a suitably chosen Kegel cover we are able to fabricate the natural module and its geometry. There are few references concerning ultraproducts which are elementary and readily available (but see [**?**, pp.64-67]). Therefore in an appendix we provide a primer on their use in the context of interest to us, the representation theory of groups.

Let $\{(G_i, N_i) \mid i \in I\}$ be a Kegel cover of the infinite locally finite simple group G. Order the index set I by declaring $i < j$ if and only if $G_i < G_j$ with $G_i \cap N_j = 1$. This ordering has the property that $i \leq k \geq j$ implies $\langle G_i, G_j \rangle \leq G_k$. Next let $\mathcal{F}$ be an ultrafilter generated by the directed set $(I, \leq)$, as described in the appendix. For each $i \in I$, let $(\varphi_i, c_i)\colon G_i \to GL_{F_i}(W_i)$ be a projective representation of G_i whose kernel is N_i. Let $W = \prod_{\mathcal{F}} W_i$ be the ultraproduct vector space over the ultraproduct field $F = \prod_{\mathcal{F}} F_i$.

(3.11) THEOREM. *The ultraproduct provides a faithful projective representation $(\varphi, c)\colon G \to GL_F(W)$. Furthermore:*

(1) *If, for each $i \in I$, the cocycle c_i is trivial, then c is trivial; that is, if each φ_i is a representation, then φ is a representation.*

(2) *If, for each $i \in I$, G_i in its action on W_i leaves invariant a nondegenerate form of type Cl, then G in its action on W leaves invariant a nondegenerate form*

of type Cl.

(3) *If, for each $i \in I$, the space W_i has F_i-dimension less than k, then W has F-dimension less than k and G is a finite dimensional linear group.*

(4) *Let $1 \neq g \in G$. If, for each $i \in I$, the commutator $[W_i, \varphi_i(\langle g \rangle \cap G_i)]$ has F_i-dimension less than k, then $[W, \varphi(g)]$ has dimension less than k. In particular, G is a finitary linear group on W which leaves invariant a nondegenerate form of type Cl if each $\varphi_i(G_i)$ does.*

PROOF. Most of this is immediate from the remarks and results presented in the appendix, particularly Theorem **??**.

For the final remark of parts (3) and (4), we must show that the existence of a faithful projective representation $(\varphi, c): G \to FGL_F(W)$ implies the existence of a corresponding genuine representation $\tilde{\varphi}$. For (3) this is immediate; there is a faithful and finite dimensional representation $\tilde{\varphi}: G \to GL(W^\star \otimes W)$. In the situation of (4) this particular representation is not finitary when W has infinite dimension, and something else must be done.

Consider now (4), and assume that W has infinite dimension. Let Z be the group of scalars in $GL_F(W)$, so that Z is isomorphic to the multiplicative group of F. As φ is a projective representation, $\varphi(G).Z$ is a subgroup of $GL_F(W)$. Here $\varphi(G).Z/Z \simeq G$ is simple. As $\varphi(G).Z \cap FGL_F(W).Z \geq \langle g, Z \rangle$, we must have $\varphi(G).Z \leq FGL_F(W).Z$. As W has infinite dimension, $Z \cap FGL_F(W) = 1$; so $FGL_F(W).Z = FGL_F(W) \times Z$. Therefore we can construct the desired representation $\tilde{\varphi}$ as the map φ followed by projection onto $FGL_F(W)$, so that the image $\tilde{\varphi}(G)$ equals $\varphi(G).Z \cap FGL_F(W)$.

If $\varphi(G)$ leaves invariant a nondegenerate form of type Cl, then $\varphi(G).Z$ acts on a 1-space of such forms. Its perfect subgroup $\tilde{\varphi}(G)$ therefore leaves each of these forms invariant. □

Theorem **??** provides us with two valuable corollaries.

(3.12) COROLLARY. *A locally finite simple group G which has a sectional cover composed of sporadic, cyclic, exceptional Lie type groups, or classical groups of bounded dimension has a faithful representation as a linear group in finite dimension.*

PROOF. For the groups belonging to these classes, there is a k such that each group has a faithful representation on a vector space of dimension of at most k. Therefore G has a faithful representation in dimension at most k by part (3) of the theorem. □

In particular **BBHST** (Theorem **??**) applies to say that G is either finite or of Lie type over a locally finite field.

Now choose $1 \neq g \in G$. If $g \in G_i$ and G_i/N_i is an alternating or classical group, let (φ_i, c_i) be a natural representation chosen to minimize $\dim_{F_i} \varphi_i(g)$, the *natural degree* of g in G_i/N_i. (If $g \notin G_i$, take the natural degree of g in G_i/N_i to be 0.) An immediate consequence of part (4) of the theorem is then:

(3.13) Corollary. *A locally finite simple group G which has a sectional cover composed of alternating groups or classical groups of unbounded dimension in which the natural degrees of the element $g \neq 1$ are bounded has a faithful representation as a finitary linear group.*

4. Around a Proof

We discuss a proof of Theorem **??**. We first show that a group as hypothesized resembles the conclusions internally, then we reconstruct its associated geometry and identify it externally.

4.1. The Attack

Let G be a locally finite simple group which is a subgroup of $FGL_K(V)$ but has no faithful representation as a linear group in finite dimension. Assume that G has the Kegel cover $\mathcal{K} = \{(G_\lambda, N_\lambda) \mid \lambda \in \Lambda\}$.

Our approach has five basic steps:

- **A.** Reconstruct the unipotent-by-quasisimple cover "$\mathcal{H} \cap \mathcal{CC}(G)$";
- **B.** Find root elements;
- **C.** Find a quasisimple cover "$\mathcal{CC}_{qs}(G)$";
- **D.** Rebuild the natural module and its geometry;
- **E.** Identify G as the isometry group generated by all eligible root elements.

We begin by restricting the cover $\mathcal{K}$ under consideration.

Choose an arbitrary but fixed nonidentity element g of G, and set $d = \deg_V g = \dim_K[V, g]$. As mentioned after Lemma **??**, the members of $\mathcal{K}$ with g in $G_\lambda - N_\lambda$ form a subcover. Discarding the unnecessary members of $\mathcal{K}$, we may assume that $g \in G_\lambda - N_\lambda$, for all λ.

We next further prune $\mathcal{K}$ in a typical application of the coloring argument, Lemma **??**. Choose a constant κ. The exact value of κ is not crucial, but we want it to be large compared to d, that is, $\kappa >> d$. In particular, to use Theorem **??** we need $\kappa > 144d^2$. Color the members (G_λ, N_λ) of the cover $\mathcal{K}$ with six colors, according to the isomorphism type of the simple quotient G_λ/N_λ:

1. Alternating Alt_{n_λ} with $\kappa < n_\lambda$;
2. Symplectic PSp_{n_λ} in odd characteristic with $\kappa < n_\lambda$;
3. Unitary PSU_{n_λ} with $\kappa < n_\lambda$;
4. Orthogonal $P\Omega_{n_\lambda}$ with $\kappa < n_\lambda$;
5. Linear PSL_{n_λ} with $\kappa < n_\lambda$;
6. Cyclic, sporadic, execeptional Lie type, classical of degree n_λ with $\kappa \geq n_\lambda$, or alternating of degree n_λ with $\kappa \geq n_\lambda$.

Take note that, as discussed earlier in Subsections **??** and **??**, we are including symplectic groups in even characteristic under the heading of orthogonal groups.

By the classification of finite simple groups, Theorem **??** (**CFSG**), we have assigned each member of the cover a color. Therefore by Lemma **??** there is a monochromatic subcover, one color class which is itself still a Kegel cover $\mathcal{S}$. As G is finitary but not linear, Corollary **??** says that the last color, class **6.**, does not give a cover.

In summary, we have a locally finite simple group $G \leq FGL_K(V)$ which is not linear, and an element $g \neq 1$ of G with $\deg_V g = \dim_K[V,g] = d$. The group G has a Kegel cover $\mathcal{S} = \{(G_i, N_i) \mid i \in I\}$ with $g \in G_i - N_i$, for all i; and the simple quotient G_i/N_i is of a fixed type, alternating or classical, one of **1.** through **5**. The degree of G_i/N_i is n_i with $n_i > \kappa$, for a fixed constant κ ($>> d$). Indeed the set $\{n_i \mid i \in I\}$ is unbounded (again by Corollary **??**).

In making these coloring arguments, we appear to have used the full strength of the classification of finite simple groups, **CFSG**. In fact it is possible to use Theorem **??** to show directly that nonlinear G must be covered by alternating or classical groups. Therefore, as mentioned before, Theorem **??** contains as much of the classification as required for the proof of Theorem **??**. On the other hand, most envisioned applications would involve at least further appeal to Theorem **??** whose proof requires more serious use of the classification.

4.2. A Unipotent-by-Quasisimple Cover: A

We first aim to prove that, starting from any Kegel cover, we can find one which looks like one of those for the groups of Theorem **??**. These covers are parabolic for classical groups and natural for alternating groups. The nearly natural modules for classical groups and generalized monomial modules for alternating groups were defined and discussed in Subsection **??**.

(4.1) Proposition. *For each i, there is a G_i-composition factor V_i in V with $\ker_{G_i} V_i \leq N_i$; and V_i is a nearly natural module for classical G_i/N_i or generalized monomial for alternating G_i/N_i.*

In particular, for a classical cover $\mathcal{S}$, the defining characteristic of the group G_i/N_i is the same as that of V. If V has characteristic 0, then $\mathcal{S}$ is an alternating cover.

Proof. The heart of (G_i, N_i) can not be in the kernel of every G_i-composition factor since the heart is perfect. Thus there is some G_i-composition factor V_i whose kernel is in N_i. If the action on V_i is primitive, then the proposition comes from Theorem **??**. In the imprimitive case we find that V_i is a generalized monomial module for G_i, as in Proposition **??**. □

For each i, now choose a genuinely natural module W_i over the field F_i:

G_i/N_i	W_i	F_i
Alt_{n_i}	$\mathbf{Q}^{n_i-1}$	$\mathbf{Q}$
$PCl_{n_i}(q_i)$	$\mathbf{F}_{q_i}^{n_i}$	$\mathbf{F}_{q_i}$

Here PCl indicates one of the projective classical groups. For each i, next choose a (projective) natural representation

$$\varphi_i\colon G_i \to GL_{F_i}(W_i)\,,$$

taking care to minimize $d_i = \deg_{W_i} \varphi_i(g)$. (For alternating covers, we have a genuine representation; the associated cocycle can be taken to be trivial.)

(4.2) Corollary. *For the projective representation $\varphi_i\colon G_i \to GL_{F_i}(W_i)$ we have*

$$d_i = \deg_{W_i} \varphi_i(g) = \dim_{F_i}[W_i, \varphi_i(g)] \le \dim[V_i, g] \le d\,.$$

Proof. By Proposition **??** the commutator $[W_i, \varphi_i(g)]$ has dimension at most that of $[V_i, g]$ which is a section of $[V, g]$ of dimension d. □

As in Theorem **??** and Corollary **??**, the simple group G acts on $\prod_{\mathcal{F}} W_i$, the ultraproduct of the spaces W_i and a vector space over the field $\prod_{\mathcal{F}} F_i$. It is convenient to extend this field ultraproduct to its algebraic closure F and the vector space ultraproduct to W, its tensor product with the field F. By the results mentioned and Corollary **??**, $\dim_F[W, g] = d_0 \le d$. Thus G acts as a subgroup of $FGL_F(W)$.

At this point it may be worth standing back and considering what we have accomplished; starting with a finitary group, we have proved that it is indeed a finitary group. This does not sound like much, but in fact we have made a great deal of progress. While the original representation had no additional structure, the new representation has been constructed according to a precise recipe and so has many built-in properties. For instance, again by Theorem **??**, if $\mathcal{S}$ has classical type PCl, then G acts on W leaving invariant a form of the same type as the members of $\mathcal{S}$. Our original space might have been highly irreducible, say, a direct sum of finitely many copies of some faithful finitary module. As is the case with the natural modules for the groups we seek, our new space must be essentially irreducible since each G_i has a unique nontrivial composition factor within it (see Theorem **??** below).

Consider now the implications of Proposition **??** and its corollary for the finitary action of G on W. We learn that each G_i has in W a composition factor whose kernel is contained in N_i and which is nearly natural for G_i/N_i. In this factor the element g has degree at most d_0. But from our choice of the φ_i, in any nearly natural representation, g has degree at least d_0. Therefore the degree is exactly d_0; there is a unique G_i-composition factor in W on which g acts nontrivially; and this factor is nearly natural for G_i/N_i. If G_i/N_i is alternating then this factor is monomial, admitting some quotient $Z_m^{n_i-1}.Alt_{n_i}$ of G_i, since the restriction on the degree of g forces any associated block of imprimitivity to have dimension 1.

We pull together some of the properties of our new finitary representation for G. As G is faithful on W, we may identify G with its image in $FGL_F(W)$.

(4.3) Theorem. *The locally finite simple group G is a subgroup of $FGL_F(W)$ with F algebraically closed. For a fixed $1 \ne g \in G$, we have $d_0 = \dim_F[W, g] << \kappa$. Let $\mathcal{S} = \{(G_i, N_i) \,|\, i \in I\}$ be a Kegel cover for G having one of the types* **1.** *through* **5.***, and such that $g \in G_i - N_i$, for all $i \in I$.*

(1) *In its action on W, each G_i has a unique nontrivial composition factor.*

(2) *If the cover $\mathcal{S}$ has alternating type as in* **1.***, then F has characteristic* 0 *and the composition factor of* (1) *is monomial.*

(3) *If the cover $\mathcal{S}$ has classical type as in one of* **2.** *through* **5.***, then F has the same characteristic p as each classical group G_i/N_i, the composition factor of* (1) *is*

nearly natural of degree n_i, and the set $\{n_i \,|\, i \in I\}$ is unbounded. In cases **2.**, **3.**, *and* **4.**, *G leaves invariant a nondegenerate form of the same type as that for the classical groups G_i/N_i.*

For each $i \in I$, let $H_i = \langle g^{G_i} \rangle$ and $M_i = H_i \cap N_i$.

(4.4) PROPOSITION. *$\mathcal{H} = \{(H_i, M_i) \,|\, i \in I\}$ is a Kegel cover of G whose factors are the same as those of $\mathcal{S}$. If the cover $\mathcal{S}$ is of classical type, then the subgroups H_i are unipotent-by-quasisimple in their action on W. If $\mathcal{S}$ is alternating, then the H_i are abelian-by-simple.*

PROOF. The first sentence is immediate from Proposition **??**.

As g acts nontrivially on only one G_i-composition factor in W, the same is true of H_i. Therefore the kernel of the action on this factor is a unipotent normal subgroup, and modulo this normal subgroup we get the action described above. In the case of an alternating cover, our space W has characteristic 0; so a unipotent group is torsion free. In a locally finite group such a group can only be trivial, so the monomial quotient group must be all of H_i. $\square$

It is not hard to go one step further. If we select those members of $\mathcal{H}$ with H_i perfect, then we actually have a Kegel cover of G which is contained in the heart of $\mathcal{S}$.

Corollary **??** provides us with a converse to Theorem **??**(4). In that theorem we saw that having an element of bounded representation degree in Kegel quotients forces a locally finite simple group to be finitary. Now Corollary **??** says that in a finitary, locally finite simple group, elements must have bounded representation degree in their Kegel quotients. A little more can be squeezed out of these observations.

(4.5) THEOREM. *A simple section of a finitary locally finite group is itself finitary.*

To prove this, first find a Kegel cover for the simple section S. Pull the members of this cover back to finite subgroups of the parent locally finite group. The arguments of Proposition **??** and its corollary go through to prove that elements of the simple section S have bounded representation degree in the Kegel quotients of the section. Therefore by Theorem **??**(4) the simple group S is finitary.

4.3. ALTERNATING GROUPS

In this section we complete a proof of the following two theorems.

(4.6) THEOREM. *A locally finite simple group which is infinite and finitary and has a Kegel cover all of whose quotients are alternating groups is isomorphic to an alternating group Alt_Ω, for some infinite set Ω.*

(4.7) THEOREM. *A locally finite simple group which is infinite and finitary in characteristic* 0 *is isomorphic to an alternating group Alt_Ω, for some infinite set Ω.*

These theorems can be found in [?]. (See also [?].) The two results are seen to be equivalent, using what we have proven earlier. Indeed, a Kegel cover of a characteristic 0 group must have alternating type by Proposition **??**. Conversely, in the previous section we constructed a faithful finitary representation in characteristic 0 for any locally finite simple and finitary group that possesses a Kegel cover of alternating type.

We isolate the alternating case from the classical because we are able to give a nearly complete proof. Although the alternating case is a little different and a little easier than the classical case, the proof given here is a good introduction to the more difficult arguments. Indeed some of the results in this section have slightly easier proofs if we use more permutation group theory, but we stay with proofs similar in spirit to those of the general case. These are, in any event, not overly deep or complex.

Assume then that we have an infinite, locally finite simple G which is finitary and has a Kegel cover whose quotients are alternating groups as in Theorem **??**(2). Thus $G \leq FGL_F(W)$ where the algebraically closed field F has characteristic 0. As in Proposition **??**, G has the Kegel cover $\mathcal{H} = \{(H_i, M_i) \mid i \in I\}$, where $H_i/M_i \simeq Alt_{n_i}$. By Maschke's theorem, $W = [W, H_i] \oplus C_W(H_i)$. The module $[W, H_i]$ is monomial for H_i. Its dimension is $n_i - \epsilon$, and the base group $M_i \simeq Z_m^{n_i-1}$ acts diagonally. (Here ϵ is 1 or 0 as $m = 1$ or $m \neq 1$.)

The element g was originally chosen as an arbitrary nonidentity element. Let us now assume that it was chosen to have odd prime order (as certainly was possible). In each representation

$$H_i \simeq M_i.Alt_{n_i} \xrightarrow{\varphi_i} GL_{F_i}(W_i) \simeq GL(\mathbf{Q}^{n_i-1}),$$

we have $\ker \varphi_i = M_i$ and $d_0 = \deg g = \dim[W_i, g] = s(p-1) << n_i$, where the element $\varphi_i(g)$ is represented by s distinct p-cycles in the natural permutation representation of $\varphi_i(H_i) \simeq Alt_{n_i}$.

We also have the injections

$$\eta_{i,j}: H_i \to H_j/M_j \simeq Alt_{n_j}$$

available to us, furnished by the Kegel cover for every $j > i$. Without further information, this faithful permutation representation of H_i could have many different types; but we also know that the element g is always represented by s distinct p-cycles and no more. (We have $d_0 = s(p-1)$; consider φ_j and the representation of H_j on W.) This additional knowledge is enough for us to kill off the Kegel kernels M_i, and to identify the injections $\eta_{i,j}$ explicitly.

(4.8) PROPOSITION. *Let the monomial group $A \simeq Z_m^{n-1}.Alt_n$ be a subgroup of Alt_Δ, and let the base group of A be $B \simeq Z_m^{n-1}$. Further assume that the element g of $A - B$ has odd prime order and is represented by s distinct p-cycles both in Alt_Δ and in the natural representation of $A/B \simeq Alt_n$. If $n >> sp$, then there is a subgroup $C \simeq Alt_n$ which contains g and complements B in A.*

PROOF. By Corollary **??**, $\bar{A} = A/B \simeq Alt_n$ is generated by at most $e = 3 + (n - 2)/s(p-1)$ conjugates of $\bar{g}$, a p-element composed of s distinct p-cycles. (One of these can be taken to be $\bar{g}$ itself). For each such conjugate $\bar{h}_j$ of $\bar{g}$, choose a preimage h_j in A which is a conjugate of g. (Take care to lift $\bar{g}$ to g.) Let C be the subgroup of A generated by the various h_j. As $\bar{C} = \bar{A}$, we have $A = BC$; so C is isomorphic to $Z_k^{n-1}.Alt_n$, for some divisor k of m. We claim that $k = 1$. Indeed C moves at most $esp = (3 + (n-2)/s(p-1))sp$ points of Δ, a number only slightly larger than n and a good deal less than $2n$ since p is odd and $n >> sp$. On the other hand, it is an easy exercise to prove that any nontrivial elementary abelian q-group Z_q^{n-1} has no faithful permutation representation of degree less than $(n-1)q$, which is at least on the order of $2n$. Thus no prime divisor q of k exists; and $k = 1$, as claimed. □

(4.9) LEMMA. *Root elements, that is, 3-cycles exist in G. More precisely, there is an element t of order 3 and a subset J of I, such that $\mathcal{H}_J = \{(H_j, M_j) \mid i \in J\} \subseteq \mathcal{H}$ is a subcover and t acts as a 3-cycle in $H_j/M_j \simeq Alt_{\Omega_j}$, for each $j \in J$.*

PROOF. Choose an $i \in I$ such that $n_i >> sp$, as in Proposition **??**, for $H_i/M_i \simeq Alt_{n_i}$. Restricting the monomial module $[W, H_i]$ to the subgroup $C_i \simeq Alt_{\Omega_i}$ guaranteed by the proposition, we have a natural module. In particular, a 3-cycle t of C_i has degree 2 on $[W, H_i]$ and so degree 2 on W. But then it also has degree 2 on $[W, H_j]$, a monomial module for H_j, for every $j \geq i$. The image of t in $Alt_{\Omega_j} \simeq H_j/M_j$ is thus also a 3-cycle, for all $j \in J = \{j \geq i\}$. □

(4.10) LEMMA. *For all $j \in J$, we have $M_j = 1$ and $H_j \simeq Alt_{\Omega_j}$.*

PROOF. For each $j \in J$ and any $j < k \in J$, the monomial group H_j is embedded isomorphically in $H_k/M_k \simeq Alt_{\Omega_k}$. Its image contains 3-cycles of H_k/M_k by Lemma **??**. Therefore, by Proposition **??**, we must have $M_j = 1$ and $H_j \simeq Alt_{\Omega_j}$. □

(4.11) LEMMA. *For all $i, j \in J$, the injection $\eta_{i,j}$ of $H_i \simeq Alt_{\Omega_i}$ with $|\Omega_i| = n_i$ into $H_j \simeq Alt_{\Omega_j}$ with $|\Omega_j| = n_j$ is induced by a unique injection of sets $\eta^*_{i,j}\colon \Omega_i \to \Omega_j$.*

PROOF. By Lemma **??**, in this embedding 3-cycles go to 3-cycles; so this is an immediate consequence of Proposition **??**. □

Now let $\Omega = \varinjlim \Omega_j$ be the direct limit of the finite sets Ω_j with respect to the injections $\eta^*_{i,j}$.

(4.12) PROPOSITION. *The group G is isomorphic to Alt_Ω.*

PROOF. The element t of Lemma **??** is a 3-cycle in each $H_j \simeq Alt_{\Omega_j}$, for $j \in J$; so it acts on Ω as a 3-cycle. For any triple α, β, γ from Ω, there is an i with $\alpha, \beta, \gamma \in \Omega_i$. Therefore the subgroup $H_i \simeq Alt_{\Omega_i}$ of G contains the 3-cycle (α, β, γ) of Alt_Ω. All possible 3-cycles are in G, so the simple group G is all of Alt_Ω. □

4.4. ROOT ELEMENTS: **B**

The previous subsection handles the alternating case (2) of Theorem **??**, so from now on we may assume we are in the classical case (3), with a Kegel cover having one of the types **2.** through **5.**

In the alternating case we found root elements by first splitting each Kegel quotient off its kernel and then using the structure of the natural module. The same is done in the classical case, starting with the following proposition which was pointed out by Ulrich Meierfrankenfeld. (The statement $E = O^p(E)$ says that E has no nontrivial homomorphic image which is a p-group.)

(4.13) PROPOSITION. *Let finite $E = O^p(E)$ act on the finite dimensional K-vector space U in characteristic p with a unique nontrivial composition factor. Assume also that $E = E_0 O_p(E)$ for $E_0 \le E$ implies $E = E_0$. Then $[E, O_p(E)] \le C_E(U)$.*

PROOF. We proceed by induction on $\dim_K U$. Set $Q = [E, O_p(E)]$.

First assume that $[U,Q]$ is not trivial as a KE-module. Then since $E = O^p(E)$, the unique nontrivial composition factor is in $[U,Q] = [U,E]$. As Q itself is unipotent, we also have $[U,Q] < U$.

Let Y be an E-invariant hyperplane of U which contains $[U,Q] = [U,E]$. By induction we have $Y \le C_U(Q)$. In particular, the action of Q on U is quadratic: $[U,Q,Q] = 0$. Choose $x \in U - Y$, so that $U = Kx \oplus Y$ and $[U,Q] = [x,Q]$ as K-space. By quadratic action the set $W = \{[x,q] \mid q \in Q\}$ is an $\mathbf{F}_p Q$-submodule of U. Indeed it is an $\mathbf{F}_p E$-submodule since $[U,E] \le C_U(Q)$, so that, for $q \in Q$ and $e \in E$,

$$[x,q]^e = [x^e, q^e] = [x + [x,e], q^e] = [x, q^e] + [[x,e], q^e] = [x, q^e] \in W\,.$$

Consider now the KE-module $\bar{U} = U/C_U(E)$. The image of $[U,E]$ is an irreducible KE-submodule $\bar{T}$ with $\bar{T} = K\bar{W}$, because $[U,E] = [x,Q]$ as a K-space. As an $\mathbf{F}_p E$-module (of possibly infinite dimension), $\bar{T}$ has a nonzero irreducible submodule $\bar{W}_0$ within finite $\bar{W}$. Thus $\bar{T} = K\bar{W} = K\bar{W}_0$ is a sum of $\mathbf{F}_p E$-irreducibles and so is completely reducible. Therefore $\bar{W}$ is complemented in $\bar{T}$; there is a $\mathbf{F}_p E$-submodule $\bar{Z}$ of $\bar{T}$ with $\bar{T} = \bar{W} \oplus \bar{Z}$.

For an arbitrary $e \in E$, we have

$$\bar{x}^e = \bar{x} + [\bar{x}, e] = \bar{x} + (\bar{w} + \bar{z}) = \bar{x} + [\bar{x}, q] + \bar{z} = \bar{x}^q + \bar{z}\,,$$

where $\bar{z}$ is in $\bar{Z}$ and $\bar{w}$ is in $\bar{W}$, so that $\bar{w} = [\bar{x}, q]$, for some $q \in Q$. Therefore $(\bar{x} + \bar{Z})^e = (\bar{x} + \bar{Z})^q$, and generally

$$(\bar{x} + \bar{Z})^E = (\bar{x} + \bar{Z})^Q\,.$$

By a Frattini argument, $E = QN_E(\bar{x} + \bar{Z})$; so by assumption $E = N_E(\bar{x} + \bar{Z})$. That is, for each $e \in E$, we have $[\bar{x}, e] \in \bar{Z}$. In particular, for each $q \in Q$, this gives $[\bar{x}, q] \in \bar{Z}$; but already $[\bar{x}, q] \in \bar{W}$. Therefore $[\bar{x}, q] \in \bar{W} \cap \bar{Z} = \bar{0}$. We conclude that $[\bar{x}, Q] = \bar{0}$, which is not true since $\bar{W}$ is nonzero. The contradiction shows that this case can not occur, and therefore $[U,Q]$ must be trivial as a KE-module.

Dually, $U/C_U(Q)$ is a trivial E-module. Therefore

$$[U,Q,E] = [E,U,Q] = 0\,,$$

whence $[Q,E,U] = 0$ by the Three Subgroups Lemma [**?**, (8.7)]. As $E = O^p(E)$, we have $[Q,E] = [O_p(E), E, E] = [O_p(E), E] = Q$; that is, Q is trivial on U. □

(4.14) COROLLARY. *For all i, H_i splits over $O_p(H_i)$.*

PROOF. Let E be a minimal supplement to $O_p(H_i)$ in H_i. In particular E is perfect. As E is finite, there is in W an E-invariant finite dimensional subspace U with $W = U \oplus C$, for some $C \leq C_W(E)$. Now E is faithful in its action on U and satisfies all the hypotheses of the proposition. Therefore perfect E intersects $O_p(H_i)$ only in a central p-subgroup. The Schur multiplier of the classical group H_i/M_i in characteristic p has trivial p-part [**?**, p.302], so this intersection is trivial. □

(4.15) PROPOSITION. *Let H be a classical group $Cl_n(q)$, for $n >> 0$, and let U be an extension of a trivial KH-module Z by a nearly natural KH-module. Then either*

(i) $U = Z \oplus [U, H]$, or

(ii) $H \simeq Sp_{2m}(q)$ with q even, and U is a nondegenerate but defective nearly natural module for $H \simeq \Omega_{2m+1}(q)$ with symplectic radical of dimension 1.

PROOF. Since the dual of a nearly natural module is also nearly natural, the proposition is equivalent [**?**] to the cohomological statement that $H^1(H, \mathbf{F}_q^n)$ is 0 but for the exceptional case *(ii)* where it has dimension 1. As such, the result is reasonably well-known; see, for instance, [**?**, Theorem 2.14] or [**?**, §1].

In fact, this result can be proven in an elementary fashion using generation results like Theorem **??** and Proposition **??**. Consider the case in which H is a unitary group with $n \geq 5$. We may assume that Z has dimension 1.

As $\bar{Y} = Y/Z$ is nearly natural, a transvection t of H has commutator of dimension 1 or 2 on Y, hence centralizer $C_Y(t)$ of codimension 1 or 2. Clearly $C_Y(t)$ contains Z, so $\bar{W} = \overline{C_Y(t)}$ has the same codimension 1 or 2 in $\bar{Y}$. The subspace $\bar{W}$ is invariant under $C_H(t)$. As $n \geq 5$, it can only be the hyperplane $C_{\bar{Y}}(t)$. Therefore $C_Y(t)$ has codimension 1, and t is a transvection on Y.

By Proposition **??**, the group H is generated by n of its transvections. Therefore $[U, H]$ has dimension at most n. But this commutator must cover the nearly natural quotient U/Z of dimension n. We conclude that $[U, H]$ has dimension exactly n, and $U = Z \oplus [U, H]$, as claimed.

This handles the unitary case, and the argument for the special linear case is essentially identical. (In large enough dimension n, these groups are generated by n transvections [**?**, Theorem 4.9].) For H symplectic in odd characterisitic, the result is trivial since $Z(H) = \{\pm 1\}$ and $[U, H] = [U, Z(H)]$ intersects Z trivially. For H symplectic in characteristic 2 and of dimension $2n > 2$, the argument given above for the unitary groups can be adapted. Transvections again act as transvections, but now (see Theorem **??**) we need $2n + 1$ transvections to generate H. The argument of the previous paragraph then suggests the (real) possibility of a nonsplit extension with a trivial submodule of dimension 1, but larger trivial submodules can be ruled out. Similar arguments for the orthogonal groups require the use of Siegel elements and so are messier. □

(4.16) THEOREM. *G contains root elements in its action on W.*

PROOF. Choose an i, and let L be a complement to $O_p(H_i)$ in H_i. (If L is symplectic in characteristic 2 so that $L \simeq Sp_{2n}(q) \simeq \Omega_{2n+1}(q)$, then replace L by an orthogonal subgroup $L_0 \simeq \Omega^{\epsilon}_{2n}(q)$.) In W there is a unique L-composition factor and that is nearly natural. Therefore $[W, L]$ is an extension of a trivial KL-module Z by this nearly natural composition factor. By the previous proposition, we have $[W, L] = Z \oplus [W, L, L]$; but L is perfect, so $[W, L] = [W, L, L]$ is nearly natural for L and $Z = 0$. Thus $W = [W, L] \oplus C_W(L)$. In the nonorthogonal cases, a transvection of L on its natural module is also a transvection on W. In the orthogonal case, a Siegel element s of L will also have commutator dimension 2 on W. As L is irreducible on nearly natural $[W, L]$, the L-invariant forms on this module are unique up to scalar multiples. Therefore this dimension 2 commutator which is singular for L is singular in the orthogonal geometry on W, and s is a Siegel element on W. Thus in all cases, the root elements of L become root elements of G on W. □

4.5. The Rest

Now that we have root elements, the end of the proof is in sight, although some of the arguments are still rather delicate in nature. We present this part in less detail.

4.5.1. *Quasisimple Complements:* **C**

Unlike the alternating case, for the classical groups there are Kegel covers which are not just modified quasisimple covers. To pave the way for the geometric reconstruction to come, we replace our given Kegel cover by a related quasisimple cover. If we think of our cover as having parabolic type, what we now wish to do is restrict our attention to Levi subgroups. From the geometric viewpoint, we want only to consider subspaces with trivial radical.

Originally the element g was chosen as an arbitrary nonidentity element. It is now convenient to go back and rechoose so that it is a root element of G. This is possible by Theorem **??**, and allows us to assume that each H is generated by root elements.

For each $i \in I$, let

$$\mathcal{L}_i = \{L \le H_i \mid L \cap O_p(H_i) = 1, L \text{ quasisimple of type } Cl\}.$$

Then set $\mathcal{L} = \cup_i \mathcal{L}_i$.

(4.17) Proposition. *$\mathcal{L}$ is a quasisimple cover of G.*

Consider the case in which $Cl = Sp$ and F has odd characteristic. For each $i \in I$, we need to find a $j \in I$ and an $L \in \mathcal{L}_j$ with $H_i \le L$. If H_i itself is quasisimple, there is nothing to prove; so we may assume that this is not the case.

Choose a j with $H_i \le H_j$ but $H_i \cap M_j = 1$. The symplectic space W is nondegenerate, and in particular $C_W(G) = 0$. Thus we may choose our j so that additionally $[W, H_i] \cap C_{[W,H_j]}(H_j) = 0$. (Some argument is needed here. This is actually a "Kegel cover" property for the irreducible G-module W.) Therefore $[W, H_i]$ is embedded isometrically in the nearly natural module $\bar{X} = [W, H_j]/C_{[W,H_j]}(H_j)$ for the symplectic group $\bar{H} = H_j/O_p(H_j) \simeq Sp_{2n}(q)$. Set $k = \dim[W, H_i] = \dim[\bar{X}, \bar{H}_i]$. As H_i is not quasisimple, we have $k < 2n$.

By Theorem **??**, there is a set $\bar{\Sigma}$ of $2n - k$ transvections $\bar{t}$ of $\bar{H}$ such that $\bar{H} = \langle \bar{H}_i, \bar{\Sigma}\rangle$. We lift each individual transvection $\bar{t}$ of $\bar{\Sigma}$ to a transvection t of H_j and call the corresponding set of $2n-k$ transvections Σ. Consider $L = \langle H_i, \Sigma\rangle$. By construction $\bar{L} = \bar{H}$ with $\bar{X} = [\bar{X}, \bar{L}]$ a nearly natural module of dimension $2n$. However

$$\dim[W, L] \le \dim[W, H_i] + \dim[W, \Sigma] \le k + (2n-k) = 2n\,.$$

We conclude that $[W, L]$ of dimension $2n$ is a nearly natural module for the quasisimple group $L \simeq \bar{H} \simeq Sp_{2n}(q)$. In particular $H_i \le L \in \mathcal{L}_j$, as required.

4.5.2. *Reconstructing the Space:* **D**

In Theorem **??** the space ${}_KV$ was replaced by the new, essentially irreducible G-module ${}_FW$; but this new space is still not ideal. The field F may not be the natural one for the group G. Indeed since F is the algebraic closure of an ultraproduct of finite fields it typically has large transcendence degree. Now we construct another new space on which G acts, this one defined over a direct limit of finite fields which is thus locally finite.

We begin with a result of Hartley and Shute:

(4.18) THEOREM. *([**?**, Theorem C.]) Let $\mathbf{F}_{q_1}$ and $\mathbf{F}_{q_2}$ be two finite fields, and let $G_1 \simeq \Phi(\mathbf{F}_{q_1})$ and $G_2 \simeq \Phi(\mathbf{F}_{q_2})$ be two quasisimple Lie type groups of the same type Φ. Then G_1 is isomorphic to a subgroup of G_2 if and only if $\mathbf{F}_{q_1}$ is isomorphic to a subfield of $\mathbf{F}_{q_2}$. In this case, any two subgroups of G_2 isomorphic to G_1 are conjugate via an automorphism of G_2.*

We are particularly interested in the case where each G_i is a classical group, $Cl_n(\mathbf{F}_{q_i})$. There is a canonical subgroup of the matrix group G_2 isomorphic to G_1, namely the corresponding matrix subgroup over the subfield $\mathbf{F}_{q_1}$ of $\mathbf{F}_{q_2}$. The embedding of G_1 as this subgroup is induced by the canonical $\mathbf{F}_{q_1}$-linear map of $X_1 \simeq \mathbf{F}_{q_1}^n$ into $X_2 \simeq \mathbf{F}_{q_2}^n$. The space X_2, as $\mathbf{F}_{q_1}G_1$ module, is a direct sum of $|\mathbf{F}_{q_1}\!:\!\mathbf{F}_{q_2}|$ isomorphic copies of X_1, permuted among themselves by $\mathbf{F}_{q_2}$ scalar multiplication.

In the cases $Cl \in \{Sp, SU, \Omega\}$, each automorphism of G_2 comes from conjugation by a matrix of $GL_{\mathbf{F}_{q_2}}(X_2)$ which normalizes G_2 followed by an automorphism of the field $\mathbf{F}_{q_2}$ (if we assume that $n > 8$ when $Cl = \Omega$). Thus in these cases, Theorem **??** can in part be rephrased in the geometric form:

(4.19) THEOREM. *Let η be an isomorphism of the quasisimple classical group $G_1 \simeq Cl_n(\mathbf{F}_{q_1})$ into $G_2 \simeq Cl_n(\mathbf{F}_{q_2})$, where $Cl \in \{Sp, SU, \Omega\}$ and $n > 8$. Then there is an isomorphism σ of $\mathbf{F}_{q_1}$ into $\mathbf{F}_{q_2}$, and a σ-semilinear injective isometry η^* of $X_1 \simeq \mathbf{F}_{q_1}^n$ into $X_2 \simeq \mathbf{F}_{q_2}^n$ which induces η. The map η^* is uniquely determined up to scalar multiplication by a member of $\mathbf{F}_{q_2}$.*

A new proof (and extension) of Hartley and Shute's Theorem **??** has been given by Liebeck and Seitz [**?**]. The basic observation is that root elements of G_1 must be mapped to root elements of G_2. To convince yourself of this, think of how the group of lower unitriangular matrices over the subfield $\mathbf{F}_{q_1}$ fits into that over the field $\mathbf{F}_{q_2}$. The center of the small subgroup (Sylow in $SL_n(\mathbf{F}_{q_1})$) falls into the center of the large subgroup (Sylow in $SL_n(\mathbf{F}_{q_2})$). But these centers are in both cases composed of transvections, root elements in the case $Cl = SL$. For any abstract embedding η of $SL_n(\mathbf{F}_{q_1})$ into $SL_n(\mathbf{F}_{q_2})$ consideration of nilpotency class shows that the center of the (small) Sylow group of the image of η must lie in the center of the large Sylow subgroup. Therefore root elements are taken by η to root elements.

The apparently small change of allowing G_1 and G_2 to have different dimensions produces drastic results. Concerning an arbitrary injection η of $G_1 \simeq Cl_{n_1}(\mathbf{F}_{q_1})$ into $G_2 \simeq Cl_{n_2}(\mathbf{F}_{q_2})$, we can say almost nothing. Indeed, from any pair of fields $\mathbf{F}_{q_1}$ and $\mathbf{F}_{q_2}$ and any degree k permutation representation of G_1, we can construct an embedding of G_1 in $Cl_k(\mathbf{F}_{q_2})$. If we now require that η respect root elements (rather than proving it along the way), then order is restored:

(4.20) THEOREM. *Let η be an isomorphism of the quasisimple classical group $G_1 \simeq Cl_{n_1}(\mathbf{F}_{q_1})$ into $G_2 \simeq Cl_{n_2}(\mathbf{F}_{q_2})$, where $Cl \in \{Sp, SU, \Omega\}$ and $n_1 > 8$. Assume additionally that, for the root element g of G_1, the element $\eta(g)$ is a root element of G_2. Then there is an isomorphism σ of $\mathbf{F}_{q_1}$ into $\mathbf{F}_{q_2}$, and a σ-semilinear injective isometry η^* of $X_1 \simeq \mathbf{F}_{q_1}^{n_1}$ into $X_2 \simeq \mathbf{F}_{q_2}^{n_2}$ which induces η. The map η^* is uniquely determined up to scalar multiplication by a member of $\mathbf{F}_{q_2}$.*

The result can be proven using the earlier result, or it can be given a direct proof from first principles since root elements carry so much information about the related geometry. As mentioned before, a classical group is generated by what is essentially the smallest possible number of root elements (see Theorem **??** and Proposition **??**). Since η respects root elements, we find that $Y = [X_2, \eta(G_1)]$ has dimension essentially n_1 and so must look something like the natural module X_1 tensored up to $\mathbf{F}_{q_2}$. This is the beginning of the construction of the semilinear map. Theorem **??** gives the action of $\eta(G_1)$ on the nondegenerate and nearly natural module Y, and the only way of extending this action to all of X_2 is by making $\eta(G_1)$ trivial on $Y^\perp$.

Some care must be taken when the n_i are both odd, the q_i are even, and $Cl = \Omega$. In this case neither X_1 nor X_2 are irreducible, and there are two fundamentally different embeddings, depending upon whether or not the symplectic radical of X_1 is mapped into that of X_2 or not.

We are now in a position to reconstruct the natural space on which our group G acts in the cases $Cl \in \{Sp, SU, \Omega\}$. For each $L \in \mathcal{L}$, we have $L \simeq Cl_{n_L}(\mathbf{F}_{q_L})$. Let $X_L \simeq \mathbf{F}_{q_L}^{n_L}$ be a natural $\mathbf{F}_{q_L}L$-module. To create our G-space X, it will be helpful to fix some member $L \in \mathcal{L}$ and prune $\mathcal{L}$ to the quasisimple cover $\mathcal{Q} = \{Q \in \mathcal{L} \mid L \le Q\}$. By the previous theorem, for each $Q \in \mathcal{Q}$, there is a field automorphism $\sigma_{L,Q}$ and a $\sigma_{L,Q}$-semilinear map $\eta^*\colon X_L \to X_Q$ which induces the inclusion $L \le Q$. This map η^* is unique up to scalar multiplication by some member of $\mathbf{F}_{q_Q}$. For each $Q \in \mathcal{Q}$, choose and fix one of these multiples $\eta^*_{L,Q}$. Next, for each pair $P, Q \in \mathcal{Q}$ with $P \le Q$, there is a collection of maps ρ which are $\sigma_{P,Q}$-semilinear maps from X_P to X_Q, inducing the inclusion $P \le Q$. Pairwise these maps ρ differ only by a scalar multiple from $\mathbf{F}_{q_Q}$. In particular, their images are $\mathbf{F}_{q_P}$-subspaces of X_Q with trivial pairwise intersections. On the other hand, the restriction of the inclusion $P \le Q$ from P to its subgroup L is the inclusion $L \le Q$. Therefore exactly one of the maps ρ will satisfy

$$\eta^*_{L,P}\rho = \eta^*_{L,Q} \,.$$

In this case, we set $\eta^*_{P,Q} = \rho$.

The injective maps $\eta^*_{P,Q}$ and $\sigma_{P,Q}$, for all $P, Q \in \mathcal{Q}$ with $P \le Q$ allow us to construct the vector space $X = \varinjlim X_Q$ over the field $E = \varinjlim \mathbf{F}_{q_Q}$. We then see that X is a finitary EG-module. We can further construct an invariant nondegenerate form of type Cl on X as the direct limit of invariant forms on the X_Q, having used a L-invariant nondegenerate form to normalize as before. (Indeed a small amount additional argument would show that E is a subfield of F and that X is isometric to an E-subspace of W.) In these cases, this completes the reconstruction **D**.

There are two related reasons why the case $Cl = SL$ must be treated differently from that of the other classical groups. First, the result corresponding to Theorem **??** must allow for the transpose-inverse automorphism; geometrically, we must allow for dualities as well as semilinear maps. The second difficulty arises because, in the proof of the result corresponding to Theorem **??**, there is no canonical complement $Z^\perp$ to $Z = [X_2, \eta(G_1)]$ available for unique extension of the action on Z. The answer to both problems is to consider G_i as acting on the direct product of the spaces X_i and $Y_i = X_i^\star$ preserving the natural nondegenerate pairing $p\colon X_i \times Y_i \to \mathbf{F}_{q_i}$ given by $p(x, y) = xy$.

(4.21) THEOREM. *Let η be an isomorphism of the quasisimple classical group $G_1 \simeq SL_{n_1}(\mathbf{F}_{q_1})$ into $G_2 \simeq SL_{n_2}(\mathbf{F}_{q_2})$ with $n_1 > 2$. Assume additionally that, for the root element (transvection) g of G_1, the element $\eta(g)$ is a root element of G_2. Set $X_i = \mathbf{F}_{q_i}^{n_i}$ and $Y_i = X_i^\star$.*

Then there is an isomorphism σ of $\mathbf{F}_{q_1}$ into $\mathbf{F}_{q_2}$, and a σ-semilinear injective isometry $\eta^*\colon X_1 \times Y_1 \to X_2 \times Y_2$ which induces η and such that either

(a) $\eta^*(X_1) \le X_2$ and $\eta^*(Y_1) \le Y_2$; or

(b) $\eta^*(X_1) \le Y_2$ and $\eta^*(Y_1) \le X_2$.

In any event the map η^* is uniquely determined up to scalar multiplication by a member of $\mathbf{F}_{q_2}$.

Part (b) describes dualities as opposed to semilinear maps. The proof of uniqueness involves the observation that the set of transvections fixing a 1-space of X_1 must either fix a 1-space of X_2 (case (a)) or a hyperplane of X_2 (case (b)), but does not fix both since $n_1 > 2$. A canonical complement to $[X_2, \eta(G_1)]$ in X_2 is now provided by $[Y_2, \eta(G_1)]^\perp$ (and similarly with the roles of X_2 and Y_2 reversed).

To find paired spaces on which G acts, we once again prune $\mathcal{L}$ down to its subcover $\mathcal{Q}$ of all members containing some fixed L. This and the theorem allow us to construct as before field maps $\sigma_{L,Q}$ and semilinear isometries $\eta^*_{L,Q}$. Here we must choose, for all Q including L, paired spaces X_Q and Y_Q admitting Q. We make our choices so that always $\eta^*_{L,Q}$ takes X_L into X_Q and Y_L into Y_Q. Continuing as before, for each pair $P, Q \in \mathcal{Q}$ with $P \le Q$, we find a scalar collection of $\sigma_{P,Q}$-semilinear maps from $X_P \times Y_P$ to $X_Q \times Y_Q$, each of which induces the inclusion $P \le Q$. Exactly one of these possible choices for $\eta^*_{P,Q}$ will make true the equation

$$\eta^*_{L,P}\eta^*_{P,Q} = \eta^*_{L,Q}\,,$$

and this is the choice we make.

Our original choices for L guarantee that always $\eta^*_{P,Q}$ takes X_P into X_Q and Y_P into Y_Q. Thus we can define the pair of spaces $X = \varinjlim X_Q$ and $Y = \varinjlim Y_Q$ over the field $E = \varinjlim \mathbf{F}_{q_Q}$. On $X \times Y$ there is an G-invariant nondegenerate pairing which is the direct limit of those on the various $X_Q \times Y_Q$. Our reconstruction is complete for the groups of type SL as well.

4.5.3. *Identifying the Group:* **E**
At the close of the previous subsection we had identified our group G as a subgroup of one of $FSp_E(X,s)$, $FSU_E(X,u)$, $F\Omega_E(X,q)$, or $T_E(Y,X)$. It remains to prove that we have equality. This is now largely a matter of book-keeping. Using the construction of X as a direct limit (and Y in the final case), we can reveal the large finitary group as a direct limit itself, and that can then be seen to be isomorphic to G, the direct limit of the Q in $\mathcal{Q}$. Alternatively we can prove that, in its action on X (and Y), the group G contains every root element of the larger group. At that point results like Theorem **??** prove the equality. The proof of Theorem **??** is then complete.

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Appendix

A PRIMER ON ULTRAPRODUCTS OF GROUPS

A. Introduction

Let G be a group and $\{N_i \mid i \in I\}$ be a set of normal subgroups. When $\bigcap_i N_i = 1$, there is a natural embedding of G in the Cartesian product $\prod_{i\in I} \bar{G}_i = \prod_i \bar{G}_i$ of the groups $G/N_i = \bar{G}_i$. This allows us to transfer many properties of the groups $\bar{G}_i$ to the group G. For instance, if each $\bar{G}_i$ has exponent bounded by e, then so does G. Of particular interest is the ability to combine modules for the individual $\bar{G}_i$ together into one large module for G.

Of course many groups G do not have this rich normal structure. Indeed in the main body of the paper we are primarily interested in simple groups G. What every group does have is subgroups, so it would be highly desirable to write G as some sort of sub-Cartesian product of certain of its subgroups G_i. This we do using the ultraproduct of groups. Ultraproducts allow us to patch together global properties of G out of local properties of the G_i.

The reader may notice that the proofs presented here have a repetitive nature. This is because the general ultraproduct is a model theoretic construct which transfers first order properties from the coordinate objects to a global object that is a quotient of the Cartesian product. As such, ultraproducts have many interesting applications outside of the group theoretic realm, but we shall not discuss them here.

We would prefer not to require information on all subgroups of G, but if not all then how large a set of subgroups $\{G_i \mid i \in I\}$ (where I is some indexing set) is needed? Before the condition $\bigcap_i N_i = 1$ was enough, but a normal subgroup N_i arrives with more luggage than an arbitrary subgroup G_i. Certainly we must have $\bigcup_i G_i = G$, but now this is not enough. Equations such as $g \cdot h = k$ must be verifiable entirely within some coordinate, so there must be an i for which all three of g, h, k belong to G_i. (For instance, if g and h are in no common G_i, then their natural images in $\prod_i G_i$ commute.) In fact we want still more. The motivating observation is that every set is the direct limit of its finite subsets, whence every group is the direct limit of its finitely generated subgroups. A suitable set $\{G_i \mid i \in I\}$ of subgroups will be one which has G as its direct limit. As defined in the main body, a set of subgroups $\{G_i \mid i \in I\}$ of G is a *local system* for G if $\bigcup_i G_i = G$ and, for each pair i, j from I, there is an k with $\langle G_i, G_j \rangle \leq G_k$. As desired, the group G is the direct limit of the members of any local system $\mathcal{C}$; so all the information describing G is to be found in the members of $\mathcal{C}$ together with their containment relations.

Starting with a local system $\mathcal{C} = \{G_i \mid i \in I\}$, we attempt to reconstruct G within $\prod_i G_i$. There is a natural "diagonal embedding" which injects G as a set into $\prod_i G_i$ via $g \mapsto g^{\mathcal{C}}$:

$$g_i^{\mathcal{C}} = g \text{ if } g \in G_i\,,$$

$$g_i^{\mathcal{C}} = 1 \text{ if } g \notin G_i\,.$$

Unfortunately we will not, in general, have

$$g^{\mathcal{C}} h^{\mathcal{C}} = (gh)^{\mathcal{C}}\,,$$

since the two sides differ at any coordinate i for which G_i contains exactly one of g and h. Still the equality is true "almost everywhere," and this is the essence of the ultraproduct construction.

B. Filters, Ultrafilters, and Ultraproducts

Let I be any nonempty set. A *filter* $\mathcal{F}$ on I is a set of subsets of I which satisfies two axioms:

1. if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
2. if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.

If the empty set is in $\mathcal{F}$, then by **2.** the filter $\mathcal{F}$ must in fact be the complete power set 2^I. To avoid 2^I, the *trivial filter*, the axiom $\emptyset \notin \mathcal{F}$ is sometimes included; but we do not make this assumption.

An example of a nontrivial filter on I is the *principal filter* $\mathcal{F}_a$ composed of all subsets of I which contain the element $a \in I$. For infinite I, the *cofinite filter* composed of all cofinite subsets of I is also nontrivial. (A subset is cofinite if its complement is finite.)

We say that the set I is *directed* by the partial order $\leq$ if, for every pair i, j of elements of I, there is a $k \in I$ with $i \leq k \geq j$. In this case define

$$\mathcal{F}(i) = \{a \in I \mid i \leq a\}\,.$$

The nontrivial filter generated by the directed set $(I, \leq)$ is then

$$\mathcal{F}_{(I,\leq)} = \{A \mid A \supseteq \mathcal{F}(i),\ \text{for some } i \in I\}\,.$$

If the filter $\mathcal{F}$ on I contains A and B with $A \cap B = \emptyset$, then $\mathcal{F}$ is the trivial filter 2^I. A filter which instead satisfies:

3. for all $A \subseteq I$, $A \in \mathcal{F}$ if and only if $I - A \notin \mathcal{F}$.

is called an *ultrafilter* and is a maximal nontrivial filter. Principal filters are ultrafilters, but the cofinite filter is not. In general the filter generated by a directed set is not an ultrafilter. (*Exercise*: all ultrafilters on finite I are principal.)

The union of an ascending chain of nontrivial filters on I is itself a nontrivial filter, so by Zorn's lemma every nontrivial filter is contained in an ultrafilter. Thus completing the cofinite filter gives us a nonprincipal ultrafilter on each infinite set I. (*Exercise*: a nonprincipal ultrafilter contains the cofinite filter.) We mildly abuse terminology by saying that any ultrafilter containing $\mathcal{F}_{(I,\leq)}$ is an ultrafilter generated by the directed set $(I, \leq)$.

The following property of ultrafilters is used often.

(B.1) LEMMA. *If $\mathcal{F}$ is an ultrafilter on I, then for any finite coloring of I there is exactly one color class which belongs to $\mathcal{F}$.*

The case of a 2-coloring is just the axiom **3.**, and the lemma follows by induction.

Now suppose that $\{G_i \mid i \in I\}$ is a set of sets (typically the underlying sets of a collection of groups, rings, fields, etc.) and that $\mathcal{F}$ is an ultrafilter on I. On the Cartesian product $\prod_i G_i$ define an equivalence relation $\sim_{\mathcal{F}}$ by

$$(x_i)_{i\in I} \sim_{\mathcal{F}} (y_i)_{i\in I} \text{ iff } \{i \in I \mid x_i = y_i\} \in \mathcal{F}\,.$$

The *ultraproduct* $\prod_{\mathcal{F}} G_i$ is then the quotient of the set $\prod_i G_i$ by the equivalence relation $\sim_{\mathcal{F}}$. As there may be many different ultrafilters, there are also different ultraproducts of the same set of sets. (*Exercise*: what happens if $\mathcal{F}$ is principal?)

As an instance of Lemma **??** we have

(B.2) LEMMA. *If each set G_i has finite cardinality at most q, then $\prod_{\mathcal{F}} G_i$ has cardinality at most q.*

PROOF. For each i, color the individual members of G_i with the colors $1, 2, \dots, q$, at most one element of G_i receiving any given color. The coordinate positions of each element $g = (g_i)_{i\in I}$ in the Cartesian product are q-colored. By Lemma **??** the element g is equivalent with respect to $\mathcal{F}$ to exactly one of the monochromatic elements, of which there are at most q. □

A more difficult exercise is to prove that an ultraproduct of arbitrary finite sets is either finite or uncountably infinite.

Extra structure on the G_i can be transferred to the ultraproduct.

(B.3) PROPOSITION. *An ultraproduct of groups is a group.*

PROOF. Since the Cartesian product $\Gamma = \prod_i G_i$ is a group, it is enough to prove that that the multiplication of Γ induces a well-defined multiplication on the ultraproduct $\Gamma_{\mathcal{F}} = \prod_{\mathcal{F}} G_i$. The associativity, identity, and inverses of $\Gamma_{\mathcal{F}}$ will then be naturally induced by those of Γ.

Let $g_1 = (g_{i1})_{i\in I}$ and $g_2 = (g_{i2})_{i\in I}$ be a pair of elements from Γ which are equivalent in $\Gamma_{\mathcal{F}}$, and let h_1 and h_2 be a second such pair. Equivalence implies that the subsets $J_g = \{j \in I \,|\, g_{j1} = g_{j2}\}$ and $J_h = \{j \in I \,|\, h_{j1} = h_{j2}\}$ both belong to $\mathcal{F}$, so by **1.** their intersection $J_g \cap J_h$ does as well. This intersection is certainly contained in $J = \{j \in I \,|\, g_{j1}h_{j1} = g_{j2}h_{j2}\}$, so $J \in \mathcal{F}$ by **2.** That is, g_1h_1 and g_2h_2 are equivalent with respect to $\mathcal{F}$; and multiplication in $\Gamma_{\mathcal{F}}$ is indeed well-defined. □

A similar result is

(B.4) Proposition. *An ultraproduct of fields is a field.*

Proof. This is a little more subtle. A Cartesian product of groups is a group, whereas a Cartesian product of fields is only a commutative ring with identity. Everything proceeds as with groups, except we must additionally check that in the ultraproduct we can invert nonzero elements. Indeed an element g is nonzero if and only if (it is represented by elements for which) the set of coordinate positions i with $g_i = 0_i$ is not in $\mathcal{F}$. But then the set of positions with $g_i \neq 0_i$ is in $\mathcal{F}$ by **3.** The element with g_i^{-1} in those positions and 0_i elsewhere is (a representative of) an inverse for g in the ultraproduct. □

We shall henceforth blur the distinction between an element of the ultraproduct and the elements of the Cartesian product which represent it. For the calculation of Proposition **??** it was enough that $\mathcal{F}$ be a filter, whereas Proposition **??** uses the full strength of the ultrafilter definition.

A typical consequence of Lemma **??** is that an ultraproduct of bounded finite fields is finite. An arbitrary ultraproduct of finite fields is either finite or has uncountable transcendence degree, by the remark which follows Lemma **??**. (*Exercise*: (*i*) prove that if, for some prime p, we have $\{i \in I \,|\, \mathrm{char}\ F_i = p\} \in \mathcal{F}$, then the ultraproduct of fields $\prod_{\mathcal{F}} F_i$ has characteristic p; (*ii*) prove that if there is no p for which (*i*) holds, then $\prod_{\mathcal{F}} F_i$ has characteristic 0.)

The forming of ultraproducts commutes with the taking of products. We leave as an *Exercise* the proof of

(B.5) Lemma. *There is a natural isomorphism between $\prod_{\mathcal{F}} A_i \times \prod_{\mathcal{F}} B_i$ and $\prod_{\mathcal{F}}(A_i \times B_i)$.*

To prove that group multiplication in the Cartesian product $\prod_i G_i$ induces a well-defined multiplication for the ultraproduct $\prod_{\mathcal{F}} G_i$, we needed to show that the individual coordinate multiplication functions $G_i \times G_i \to G_i$ induce a well-defined global function. Suppose more generally that, for each $i \in I$, we have a set map $\varphi_i\colon A_i \to Z_i$, where A_i and Z_i are arbitrary sets. The φ_i become the coordinate functions of a map $\varphi\colon \prod_i A_i \to \prod_i Z_i$. Assume $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ are equivalent in $\prod_i A_i$ with respect to the ultrafilter $\mathcal{F}$, and set $J = \{i \in I \,|\, x_i = y_i\} \in \mathcal{F}$. Then $K = \{i \in I \,|\, \varphi_i(x_i) = \varphi_i(y_i)\}$ contains J and so also belongs to $\mathcal{F}$. That is, in $\prod_i Z_i$ the two elements $\varphi(x)$ and $\varphi(y)$ are equivalent with respect to $\mathcal{F}$. Therefore the map $\varphi\colon \prod_i A_i \to \prod_i Z_i$ induces a well-defined map from $\prod_{\mathcal{F}} A_i$ to $\prod_{\mathcal{F}} Z_i$. We denote this map by $\varphi_{\mathcal{F}}$.

As with group multiplication, many properties of algebraic objects are described in terms of the behavior of certain maps. For instance, the fact that the abelian group M_i is a module for the ring R_i is equivalent to certain properties of the function $\varphi_i\colon M_i \times R_i \to M_i$ given by $\varphi_i(m,r) = mr$. The coordinate maps φ_i can now be sewn together to produce an ultraproduct map $\varphi_{\mathcal{F}}$, and properties which can be verified in individual coordinates are lifted to the full ultraproduct. We discover that the abelian group $M = \prod_{\mathcal{F}} M_i$ is a module for the ring $R = \prod_{\mathcal{F}} R_i$.

We can iterate this observation. If each V_i is a vector space over the field F_i, then $V = \prod_{\mathcal{F}} V_i$ is a vector space over $F = \prod_{\mathcal{F}} F_i$. If indeed V_i is an F_iG_i-module, for some group G_i, then V is an FH-module where $H = \prod_{\mathcal{F}} G_i$. We restate this in the language of representation theory.

(B.6) THEOREM. *Let I be an index set and $\mathcal{F}$ an ultrafilter on I. For each $i \in I$, let $\varphi_i\colon G_i \to GL_{F_i}(V_i)$ be a representation. Then $\varphi_{\mathcal{F}}\colon \prod_{\mathcal{F}} G_i \to GL_F(V)$ is a representation, where $F = \prod_{\mathcal{F}} F_i$ and $V = \prod_{\mathcal{F}} V_i$.* $\square$

C. Mal'cev's Representation Theorem

We are now ready to combine remarks from the previous two sections. Let G be a group and $\mathcal{C} = \{G_i \,|\, i \in I\}$ a local system for G. Let the index set I be given a direct ordering so that $i \le k \ge j$ implies $\langle G_i, G_j\rangle \le G_k$. We say that such a directed set is *compatible* with the local system $\mathcal{C}$. There may be many compatible ways of directing I. The most obvious is "$i \le k$ if and only if $G_i \le G_k$", but a slightly different order is used for the proof of Theorem **??** in the main body of the paper.

Let $\mathcal{F}$ be an ultrafilter generated by the directed set $(I, \le)$.

(C.1) THEOREM. *The injection $g \mapsto g^{\mathcal{C}}$ provides an isomorphism of G into $\prod_{\mathcal{F}} G_i$.*

PROOF. The ultraproduct formalizes our earlier statement that $g^{\mathcal{C}}h^{\mathcal{C}}$ and $(gh)^{\mathcal{C}}$ agree "almost everywhere." They agree on a member J of $\mathcal{F}$ and so are equal in the ultraproduct. Indeed, if $g \in G_a$ and $h \in G_b$, then $(g^{\mathcal{C}})_i = g$ for $i \in \mathcal{F}(a) \in \mathcal{F}$ and $(h^{\mathcal{C}})_i = h$ for $i \in \mathcal{F}(b) \in \mathcal{F}$. Therefore $(g^{\mathcal{C}})_i(h^{\mathcal{C}})_i = gh$ for $i \in \mathcal{F}(a) \cap \mathcal{F}(b) \in \mathcal{F}$. On the other hand, for $i \in \mathcal{F}(a) \cap \mathcal{F}(b)$ we have $g \in G_i \ge G_a$ and $h \in G_i \ge G_b$. Therefore $gh \in G_i$, and we have $(gh)_i^{\mathcal{C}} = gh$ for every $i \in \mathcal{F}(a) \cap \mathcal{F}(b) \in \mathcal{F}$. We may take $J = \mathcal{F}(a) \cap \mathcal{F}(b)$. $\square$

Combining this theorem with Theorem **??**, we have

(C.2) THEOREM. *For each $i \in I$, let $\varphi_i\colon G_i \to GL_{F_i}(V_i)$ be a representation. Then $\Phi_{\mathcal{F}}\colon G \to GL_F(V)$ is a representation, where $F = \prod_{\mathcal{F}} F_i$, $V = \prod_{\mathcal{F}} V_i$, and $\Phi_{\mathcal{F}}$ is the restriction of $\varphi_{\mathcal{F}}$ to G, $\Phi_{\mathcal{F}} = \varphi_{\mathcal{F}}|_G$.* $\square$

Exercise: the element $g \in G$ is in $\ker(\Phi_{\mathcal{F}})$, the kernel of $\Phi_{\mathcal{F}}$, precisely when $\{i \in I \,|\, g \in \ker(\varphi_i)\} \in \mathcal{F}$.

We have already seen that the basic defining properties of the coordinate algebraic objects are transferred to the ultraproduct. The rules which check the existence of an inverse in a field or define the action of a module are stated in terms of finite subsets of the object under concern and their relationships. Their validity in individual coordinates breeds their validity in the ultraproduct. A more specialized case of this is:

(C.3) Theorem. *Let $\mathcal{F}$ be an ultrafilter on the index set I, and let k be an integer. If, for each $i \in I$, the dimension of the vector space V_i over the field F_i is k, then the dimension of $V = \prod_{\mathcal{F}} V_i$ over $F = \prod_{\mathcal{F}} F_i$ is k.*

Proof. Subsets of the Cartesian product $\prod_i V_i$ of size n larger than k are linearly dependent over the ring $\prod_i F_i$. Consider $v_j = (v_{ij})_{i \in I} \in \prod_i V_i$, for $1 \leq j \leq n$. Choose $\alpha_j = (\alpha_{ij})_{i \in I}$ so that, for each i, $\sum_j \alpha_{ij} v_{ij} = 0$ is a nontrivial linear dependence. We claim that not every α_j is equivalent to 0 with respect to $\mathcal{F}$. Let $I_j = \{i \in I \,|\, \alpha_{ij} = 0\}$. If each I_j is in $\mathcal{F}$, then $\bigcap_j I_j$ is nonempty by **1.** and, for i in this intersection, $\sum_j \alpha_{ij} v_{ij} = 0$ is a trivial linear dependence, against construction. This proves the claim. Therefore any n elements v_j of V are linearly dependent over F, and $\dim_F V$ is at most k.

Now let $v_j = (v_{ij})_{i \in I}$, for $1 \leq j \leq k$, be elements of $\prod_i V_i$ with $\{v_{ij} \,|\, 1 \leq j \leq k\}$ linearly independent, for each i. We claim that the v_j represent F-linearly independent elements of the ultraproduct V.

Suppose that

$$\sum_{j=1}^{k} \alpha_j v_j = 0_V \,,$$

for $\alpha_j = (\alpha_{ij})_i \in F$. That is, $\sum_{j=1}^k \alpha_j v_j \sim_{\mathcal{F}} 0 \in \prod_i V_i$, for $\alpha_j = (\alpha_{ij})_{i \in I} \in \prod_{i \in I} F_i$. Thus, for some $K \in \mathcal{F}$ and all $i \in K$, $\sum_{j=1}^k \alpha_{ij} v_{ij} = 0 \in V_i$. By linear independence, $\alpha_{ij} = 0 \in F_i$, for all $i \in K$ and all $1 \leq j \leq k$. That is, α_j and 0 as elements of $\prod_i V_i$ agree in the coordinate positions of $K \in \mathcal{F}$. They are therefore equal in the ultraproduct; and $\alpha_j = 0_F$, for $1 \leq j \leq k$, as required. $\square$

Generally in the ultraproduct construction, any first order sentence will hold globally provided it holds often enough locally, that is, on some member of $\mathcal{F}$. See, for instance, the exercises which follow Proposition **??** and Theorem **??**. (*Exercise*: prove the previous theorem under the weaker hypothesis $\{i \in I \,|\, \dim_{F_i} V_i = k\} \in \mathcal{F}$.)

Clearly a group which is linear in dimension k has local systems of subgroups with each subgroup linear in dimension k. Take, for instance, the system of all finitely generated subgroups. As a consequence of Theorems **??** and **??**, we now have Mal'cev's surprising converse.

(C.4) Theorem. (Mal'cev's Representation Theorem.) *Let G be a group with a local system of subgroups each of which has a faithful representations in dimension k. Then G has a faithful representation in dimension k.* $\square$

D. Refinements

While Mal'cev's Representation Theorem **??** is striking and beautiful, much of its power stems from our ability to extend and refine it. Here we offer certain modifications and additions of specific use to us. The proofs are left as *Exercises*.

Let G be a group with local system $\{G_i \mid i \in I\}$; and, for each $i \in I$, let $\varphi_i: G_i \to GL_{F_i}(W_i)$ be a map. In general, for a map $\rho: A \to GL(X)$, the commutator $[X, A]$ is that subspace of X which is spanned by the images of all maps $\rho(a) - 1$, as a runs through A.

Give I a direct ordering which is compatible with the local system, and let $\mathcal{F}$ be an ultrafilter generated by the directed set $(I, \leq)$. Define $W = \prod_{\mathcal{F}} W_i$, a vector space over the field $F = \prod_{\mathcal{F}} F_i$. We identify G with its image in $\prod_{\mathcal{F}} G_i$, as described in Theorem **??**. Thus, as seen above, the map $\Phi_{\mathcal{F}} = (\prod_{\mathcal{F}} \varphi_i)|_G$ takes G into $GL_F(W)$.

D.1. Controlling Subgroups

Let $H \leq G$ be a subgroup of G. If, for each $i \in I$, we set $H_i = H \cap G_i$, then $\{H_i \mid i \in I\}$ is a local system for H.

(D.1) Theorem. *Suppose that, for each $i \in I$, there is a subspace U_i with $[W_i, H_i] \leq U_i \leq W_i$ and $\dim_{F_i} U_i = k$. Then $[W, H] \leq U = \prod_{\mathcal{F}} U_i$, an F-subspace of W having dimension k.*

The case $G = H$ and $W_i = U_i$ gives Mal'cev's Theorem again. If instead $H = \langle g \rangle$, then we learn that any g of bounded local degree acts as a finitary linear transformation from $FGL_F(W)$, even when W itself has infinite dimension.

D.2. Invariant Forms

Many of the classical groups are isometry groups of certain forms. The ultraproduct of isometry groups becomes an isometry group in its action on the ultraproduct.

(D.2) Proposition. *If, on each W_i, there is a quadratic or sesquilinear form of some specific type which is left invariant by $\varphi_i(G_i)$, then on W there is a $\varphi(G)$-invariant form of the same type.*

So, for instance, if each φ_i has its image in $Sp_{F_i}(W_i, f_i)$, then the form $f_{\mathcal{F}} = \prod_{\mathcal{F}} f_i$ is a symplectic form from $W \times W$ to F and is $\varphi(G)$-invariant. Furthermore, f is nondegenerate if the individual f_i are.

D.3. Projective Representations

In our applications we need a representation theorem which starts with projective representations φ_i, that is, homomorphisms into projective groups $PGL_{F_i}(V_i)$, since the natural representations of the classical simple groups are projective representations. We define projective representation in a different but equivalent form. The map $\varphi: G \to GL_F(V)$ with associated cocycle $c: G \times G \to F$ is a *projective representation* provided, for all $g, h \in G$,

$$\varphi(g)\varphi(h) = c(g,h)\varphi(gh)\,.$$

Thus a projective representation whose cocycle is identically 1 is a representation in the usual sense, a "genuine" representation. As a consequence of this definition, the cocycle c is characterized by the property:

$$c(g,h)c(gh,k) = c(g,hk)c(h,k), \text{ for all } g,h,k \in G\,.$$

(*Exercise*: verify this.) The *kernel* of the projective representation φ is given by

$$\ker(\varphi) = \{g \in G \mid \varphi(g) \text{ is scalar on } V\}\,.$$

The more general theory of projective representations can be developed in terms of modules with action "twisted" by the cocycle c. (Beware: this is not the study of what are usually called projective modules.) If we had done that earlier, then we would have found in place of Theorem **??** the result that an ultraproduct of projective representations is a projective representation whose associated cocycle is the ultraproduct of the coordinate cocycles. We then find a projective version of Theorem **??** which can be further refined by the results of the previous two subsections.

(D.3) THEOREM. *For each $i \in I$, let $(\varphi_i, c_i)\colon G_i \to GL_{F_i}(V_i)$ be a projective representation. Then $(\Phi_{\mathcal{F}}, c_{\mathcal{F}})\colon G \to GL_F(V)$ is a projective representation, where $c_{\mathcal{F}} = \prod_{\mathcal{F}} c_i$, $F = \prod_{\mathcal{F}} F_i$, $V = \prod_{\mathcal{F}} V_i$, and $\Phi_{\mathcal{F}} = (\prod_{\mathcal{F}} \varphi_i)|_G$. The element $g \in G$ is in $\ker(\Phi_{\mathcal{F}})$ if and only if $\{i \in I \mid g \in \ker(\varphi_i)\} \in \mathcal{F}$.*

(1) *If, for each $i \in I$, the dimension $\dim_{F_i} W_i$ is at most k, then $\dim_F W$ is at most k.*

(2) *If, for some $g \in G$ and each $i \in I$, the dimension $\dim_{F_i}[W_i, \varphi_i(\langle g\rangle \cap G_i)]$ is at most k, then $\dim_F[W, \Phi_{\mathcal{F}}(g)]$ is at most k.*

(3) *If each W_i has a $\varphi_i(G_i)$-invariant nondegenerate form of type Cl, then on W there is a $\Phi_{\mathcal{F}}(G)$-invariant nondegenerate form of type Cl.*

Since the ultrafilter $\mathcal{F}$ is generated by the compatible directed set $(I, \leq)$, the statement of (2) can be simplified to: if, for some $g \in G$ and each i with $g \in G_i$, the dimension $\dim_{F_i}[W_i, \varphi_i(g)]$ is at most k, then $\dim_F[W, \Phi_{\mathcal{F}}(g)]$ is at most k.