THE STRUCTURE OF RANK 3 PERMUTATION MODULES FOR $O_{2n}^{\pm}(2)$ AND $U_m(2)$ ACTING ON NONSINGULAR POINTS

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Abstract. We study the odd-characteristic structure of permutation modules for the rank 3 natural actions of $O_{2n}^{\pm}(2)$ $(n \geq 3)$ and $U_m(2)$ $(m \geq 4)$ on nonsingular points of their standard modules.

1. Introduction

Let G be either a symplectic, an orthogonal, or a unitary group. The primitive rank 3 permutation representations of G have been classified in [KL]. As a result, the natural action of G on the set of singular points (by points we mean 1-dimensional subspaces) of its standard module is always rank 3 and the associated permutation module has been studied in many papers (see [LST, L1, L2, ST]). On the other hand, the action of G on the set of nonsingular points is rank 3 if and only if G is orthogonal groups $O_{2n}^{\pm}(2)$ with $n \geq 3$ or unitary groups $U_m(2)$ with $m \geq 4$. The purpose of this paper is to describe the odd-characteristic structure, including the composition factors and submodule lattices, of the permutation modules for these groups acting on nonsingular points.

We partly utilize the notations and methods from [L1] and [ST]. Let $\mathbb{F}$ be an algebraically closed field of cross characteristic ℓ (i.e., ℓ is different from the characteristic of the underlying field of G). If the action of G on a set Ω is rank 3 then the $\mathbb{F}G$-module $\mathbb{F}\Omega$ has two special submodules, which are so-called *graph submodules* in the terminology of Liebeck [L1]. Similar to the study of cross-characteristic permutation modules on singular points, these graph submodules in our problem are *minimal* in an appropriate sense (see Propositions 3.1, 4.1, 5.1, and 6.1). When the graph submodules are different, their direct sum will be a submodule of codimension 1 in the permutation module and therefore the structure can be determined without significant effort. When they are the same (that is when char($\mathbb{F}$) = 3, as you will see later on), the problem becomes more complicated. We will look at both module and character points of view to handle this case.

The complete description of submodule structures of the permutation modules is given in the following theorem.

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Theorem 1.1. *Let $\mathbb{F}$ be an algebraically closed field of odd characteristic ℓ. Let G be either $O^{\pm}_{2n}(2)$ with $n \geq 3$ or $U_m(2)$ with $m \geq 4$ and P be the set of nonsingular points of its standard module. Then the $\mathbb{F}G$-permutation module $\mathbb{F}P$ of G acting naturally on P has the structure as described in Tables 1,2,3, and 4, where the socle series, submodule lattices, and dimensions of composition factors are determined. In these tables, $\delta_{i,j} = 1$ if $i \mid j$ and 0 otherwise. In table 2 where $G = O^{-}_{2n}(2)$, δ is the nontrivial $\mathbb{F}G$-module of dimension 1.*

The paper is organized as follows. In the next section, we collect some standard results of rank 3 permutation modules which will be used frequently later on. We also establish some relations between the actions of G on singular points and nonsingular points in §2. Each of the families of groups $O^{+}_{2n}(2)$, $O^{-}_{2n}(2)$, $U_{2n}(2)$, and $U_{2n+1}(2)$ is treated individually in §3, §4, and §5, respectively. Proof of the main theorem for each family is at the end of corresponding section. Since proofs are similar in many places, we only give detailed arguments for the case $G = O^{+}_{2n}(2)$.

Notation: Throughout the paper, $O^{\pm}_{2n}(2)$ and $U_m(2)$ are full orthogonal and unitary groups, respectively. Also, $SO^{\pm}_{2n}(2)$ and $SU_m(2)$ are special orthogonal and unitary groups. The commutator subgroup of $O^{\pm}_{2n}(2)$ is denoted by $\Omega^{\pm}_{2n}(2)$. If G is a group, $\mathrm{Irr}(G)$ (resp. $\mathrm{IBr}_\ell(G)$) will be the set of irreducible complex (resp. ℓ-Brauer) characters of G. We denote by $\overline{\chi}$ the reduction modulo ℓ of a complex character χ. If $\varphi \in \mathrm{IBr}_\ell(G)$ and λ is a constituent of φ with multiplicity k, then we sometimes say that φ has k constituents λ.

2. Preliminaries on rank 3 permutation modules

We start this section by recalling some results of rank 3 permutation modules from [Hi] and [L1].

Let G be a permutation group of rank 3 acting on the set Ω. Then, for each $\alpha \in \Omega$, G_α, which is the stabilizer of α, acts on Ω with 3 orbits $\{\alpha\}, \Delta(\alpha)$, and $\Phi(\alpha)$. We choose the notation so that $\Delta(\alpha)g = \Delta(\alpha g)$ and $\Phi(\alpha)g = \Phi(\alpha g)$. Define the following parameters associated with the action of G on Ω:

$$a = |\Delta(\alpha)|, b = |\Phi(\alpha)|,$$

$$r = |\Delta(\alpha) \cap \Delta(\beta)| \text{ for } \beta \in \Delta(\alpha),$$

$$s = |\Delta(\alpha) \cap \Delta(\gamma)| \text{ for } \gamma \in \Phi(\alpha).$$

These parameters do not depend on the choices of α, β, and γ.

Let $\mathbb{F}$ be a field of characteristic ℓ and $\mathbb{F}\Omega$ the associated permutation $\mathbb{F}G$-module. For any subset Δ of Ω, we denote by $[\Delta]$ the element $\Sigma_{\delta\in\Delta}\delta$ of $\mathbb{F}\Omega$. Set $S(\mathbb{F}\Omega) = \{\sum_{\omega\in\Omega} a_\omega \omega \mid a_\omega \in \mathbb{F}, \sum a_\omega = 0\}$ and $T(\mathbb{F}\Omega) = \{c[\Omega] \mid c \in \mathbb{F}\}$. Note that $S(\mathbb{F}\Omega)$ and $T(\mathbb{F}\Omega)$ are $\mathbb{F}G$-submodules of $\mathbb{F}\Omega$ of dimensions $|\Omega| - 1, 1$, respectively. Moreover,

Table 1. Submodule structure of $\mathbb{F}O^{+}_{2n}(2)$-module $\mathbb{F}P$.

Conditions on ℓ and n	Structure of $\mathbb{F}P$
$\ell \neq 2,3; \ell \nmid (2^n-1)$	$\mathbb{F} \oplus X \oplus Y$
$\ell \neq 2,3; \ell \mid (2^n-1)$	$\mathbb{F}$ | $X \quad \oplus \quad Y$ | $\mathbb{F}$
$\ell = 3; n$ even	$\mathbb{F} \qquad X$ \\ / Z / \\ $\mathbb{F} \qquad X$
$\ell = 3; n$ odd	X | $\mathbb{F} \quad \oplus \quad Z$ | X

where, $\dim X = \frac{(2^n-1)(2^{n-1}-1)}{3}$, $\dim Y = \frac{2^{2n}-4}{3} - \delta_{\ell,2^n-1}$,
and $\dim Z = \frac{(2^n-1)(2^{n-1}+2)}{3} - 1 - \delta_{\ell,2^n-1}$.

$T(\mathbb{F}\Omega)$ is isomorphic to the one-dimensional trivial module. We define a natural inner product on $\mathbb{F}\Omega$ by

$$\Big\langle \sum_{\omega\in\Omega} a_\omega \omega, \sum_{\omega\in\Omega} b_\omega \omega \Big\rangle = \sum_{\omega\in\Omega} a_\omega b_\omega.$$

It is easy to see that $\langle .,. \rangle$ is non-singular and G-invariant. By this reason, $\mathbb{F}\Omega$ is a self-dual $\mathbb{F}G$-module. If U is a submodule of $\mathbb{F}\Omega$, we denote by $U^\perp$ the submodule of $\mathbb{F}\Omega$ consisting of all elements orthogonal to U.

For any element $c \in \mathbb{F}$, let U_c be the $\mathbb{F}G$-submodule of $\mathbb{F}\Omega$ generated by all elements of the form $v_{c,\alpha} = c\alpha + [\Delta(\alpha)], \alpha \in \Omega$ and U'_c be the $\mathbb{F}G$-submodule of U_c generated by all elements $v_{c,\alpha} - v_{c,\beta} = c(\alpha-\beta) + [\Delta(\alpha)] - [\Delta(\beta)], \alpha, \beta \in \Omega$. It is obvious that U'_c is always contained in $S(\mathbb{F}\Omega)$. The following lemma tells that $U'_c = S(\mathbb{F}\Omega)$ for most of c.

Lemma 2.1 ([L1]). *If c is not a root of the quadratic equation*

$$x^2 + (r-s)x + s - a = 0, \tag{2.1}$$

then $U'_c = S(\mathbb{F}\Omega)$. Moreover, if c and d are roots of this equation then $\langle v_{c,\alpha}, v_{d,\beta} \rangle = s$ for any $\alpha, \beta \in \Omega$. Consequently, $\langle U'_c, U_d \rangle = \langle U'_d, U_c \rangle = 0$.

Table 2. Submodule structure of $\mathbb{F}O_{2n}^{-}(2)$-module $\mathbb{F}P$.

Conditions on ℓ and n	Structure of $\mathbb{F}P$
$\ell \neq 2, 3; \ell \nmid (2^n + 1)$	$\mathbb{F} \oplus X \oplus Y$
$\ell \neq 2, 3; \ell \mid (2^n + 1)$	$\begin{array}{ccc} & & \mathbb{F} \\ & & \mid \\ X & \oplus & Y \\ & & \mid \\ & & \mathbb{F} \end{array}$
$\ell = 3; n$ even	$\begin{array}{ccccc} & & & X & \\ & & \diagup & & \diagdown \\ \mathbb{F} \quad \oplus & \delta & & & Z \\ & & \diagdown & & \diagup \\ & & & X & \end{array}$
$\ell = 3; n$ odd	$\begin{array}{ccc} \mathbb{F} & & X \\ \mid & \diagup & \mid \\ Z & & \delta \\ \mid & \diagdown & \mid \\ \mathbb{F} & & X \end{array}$

where, $\dim X = \frac{(2^n+1)(2^{n-1}+1)}{3} - \delta_{3,\ell}$, $\dim Y = \frac{2^{2n}-4}{3} - \delta_{\ell,2^n-1}$, and $\dim Z = \frac{(2^n+1)(2^{n-1}-2)}{3} - 1 + \delta_{\ell,2^n-1}$.

Define a linear transformation on $\mathbb{F}\Omega$ as follows

$$\begin{array}{rccc} T: & \mathbb{F}\Omega & \to & \mathbb{F}\Omega \\ & \alpha & \mapsto & [\Delta(\alpha)]. \end{array} \tag{2.2}$$

It is easy to see that T is an $\mathbb{F}G$-homomorphism. Let c, d are two roots of equation (2.1). We have

$$T(v_{c,\alpha}) = c[\Delta(\alpha)] + \sum_{\delta \in \Delta(\alpha)} [\Delta(\delta)] = c[\Delta(\alpha)] + a\alpha + r[\Delta(\alpha)] + s[\Phi(\alpha)].$$

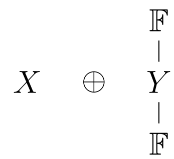
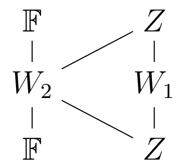
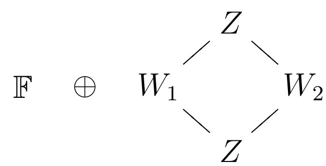
Therefore,

$$\begin{aligned} T(v_{c,\alpha} - v_{c,\beta}) &= (a - s)(\alpha - \beta) + (r - s + c)([\Delta(\alpha)] - [\Delta(\beta)]) \\ &= -cd(\alpha - \beta) - d([\Delta(\alpha)] - [\Delta(\beta)]) = -d(v_{c,\alpha} - v_{c,\beta}). \end{aligned}$$

Thus, for any $v \in U_c'$, $T(v) = -dv$. Similarly, for any $v \in U_d'$, $T(v) = -cv$. U_c' and U_d' are called graph submodules of the permutation module $\mathbb{F}\Omega$.

Now we study more details about the permutation modules for $O_{2n}^{\pm}(2)$ or $U_m(2)$ acting on nonsingular points. Let G be either $O_{2n}^{\pm}(2)$ or $U_m(2)$. Let P and P^0 be the sets of nonsingular and singular points, respectively, of the standard module

Table 3. Submodule structure of $\mathbb{F}U_{2n}(2)$-module $\mathbb{F}P$.

Conditions on ℓ and n	Structure of $\mathbb{F}P$
$\ell \neq 2,3; \ell \nmid (2^{2n}-1)$	$\mathbb{F} \oplus X \oplus Y$
$\ell \neq 2,3; \ell \mid (2^{2n}-1)$	
$\ell = 3; 3 \mid n$	
$\ell = 3; 3 \nmid n$	

where, $\dim X = \frac{(2^{2n}-1)(2^{2n-1}+1)}{9}$, $\dim Y = \frac{(2^{2n}+2)(2^{2n}-4)}{9} - \delta_{\ell,2^{2n}-1}$, $\dim W_1 = \frac{2^{2n}-1}{3}$, $\dim W_2 = \frac{(2^{2n}-1)(2^{2n-1}+1)}{9} - 1 - \delta_{3,n}$, and $\dim Z = \frac{(2^{2n}-1)(2^{2n-1}-2)}{9}$.

associated with G. Define

$$\begin{array}{rccc} Q: & \mathbb{F}P & \to & \mathbb{F}P^0 \\ & \alpha & \mapsto & [\Lambda(\alpha)] \end{array} \tag{2.3}$$

and

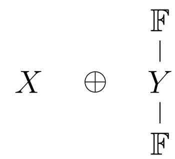
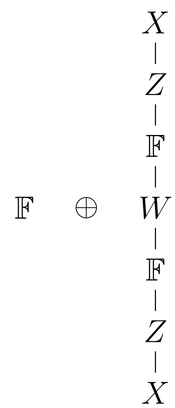
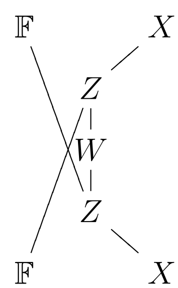
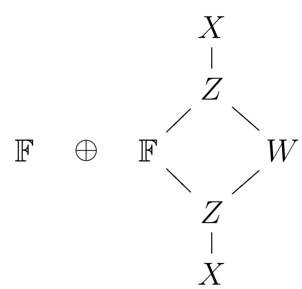
$$\begin{array}{rccc} R: & \mathbb{F}P^0 & \to & \mathbb{F}P \\ & \alpha & \mapsto & [\Gamma(\alpha)], \end{array} \tag{2.4}$$

where $\Lambda(\alpha)$ is the set of all singular points orthogonal to $\alpha \in P$ and $\Gamma(\alpha)$ is the set of all nonsingular points orthogonal to $\alpha \in P^0$. It is clear that Q and R are $\mathbb{F}G$-homomorphisms. Moreover, $\mathrm{Im}(Q|_{\mathbb{F}P}) \neq 0, T(\mathbb{F}P^0)$ and $\mathrm{Im}(R|_{\mathbb{F}P^0}) \neq 0, T(\mathbb{F}P)$. We have proved the following lemma which is a very important connection between structures of $\mathbb{F}P$ and $\mathbb{F}P^0$.

Lemma 2.2. *There exists a nonzero submodule of $\mathbb{F}P^0$ which is not $T(\mathbb{F}P^0)$ so that it is isomorphic to a quotient of $\mathbb{F}P$. Similarly, there exists a nonzero submodule of $\mathbb{F}P$ which is not $T(\mathbb{F}P)$ so that it is isomorphic to a quotient of $\mathbb{F}P^0$. Moreover, these statements are still true if $\mathbb{F}P$ and $\mathbb{F}P^0$ are replaced by $S(\mathbb{F}P)$ and $S(\mathbb{F}P^0)$, respectively.*

Let ρ and ρ^0 be complex permutation characters of G acting on $\mathbb{F}P$ and $\mathbb{F}P^0$, respectively. Since these actions is rank 3, it is well-known that both ρ and ρ^0 have

Table 4. Submodule structure of $\mathbb{F}U_{2n+1}(2)$-module $\mathbb{F}P$.

Conditions on ℓ and n	Structure of $\mathbb{F}P$
$\ell \neq 2, 3; \ell \nmid (2^{2n+1}+1)$	$\mathbb{F} \oplus X \oplus Y$
$\ell \neq 2, 3; \ell \mid (2^{2n+1}+1)$	
$\ell = 3; 3 \mid n$	
$\ell = 3; n \equiv 1 \pmod 3$	
$\ell = 3; n \equiv 2 \pmod 3$	

where, $\dim X = \frac{(2^{2n+1}+1)(2^{2n}-1)}{9}$, $\dim Y = \frac{(2^{2n+1}-2)(2^{2n+1}+4)}{9} - \delta_{\ell,2^{2n+1}+1}$,
$\dim Z = \frac{2^{2n+1}-2}{3}$, and $\dim W = \frac{(2^{2n+1}+1)(2^{2n}-4)}{9} - \delta_{3,n}$.

3 constituents, all of multiplicity 1 and exactly one of them is the trivial character. This and Lemma 2.2 imply the following:

Lemma 2.3. *ρ and ρ^0 have a common constituent which is not trivial.*

Finally, we record here a basic property of self-dual modules over a group algebra.

Lemma 2.4. *If U is a self-dual $\mathbb{F}G$-module having a self-dual, simple socle X, then both the head of U and the top layer of the socle series of U are isomorphic to X.*

Proof. This follows from Lemmas 8.2 and 8.4 of [La]. □

3. The orthogonal groups $O^{+}_{2n}(2)$

Let V be a vector space of dimension $2n \geq 6$ over the field of 2 elements $\mathbb{F}_2 = \{0, 1\}$. Let $Q(\cdot)$ be a quadratic form on V of type $+$ and $(\cdot, \cdot)$ be the non-degenerate symmetric bilinear form on V associated with Q so that $Q(au + bv) = a^2Q(u) + b^2Q(v) + ab(u, v)$ for any $a, b \in \mathbb{F}_2, u, v \in V$. Then $G = O^{+}_{2n}(2)$ is the orthogonal group of linear transformations of V preserving Q.

We choose a basic of V consisting of vectors $\{e_1, ..., e_n, f_1, ..., f_n\}$ so that $Q(e_i) = Q(f_j) = 0$ and $(e_i, f_j) = \delta_{ij}$ for all $i, j = 1, ..., n$. Let P be the set of all nonsingular points in V. Then $P = \{\langle \sum_1^n (a_ie_i + b_if_i)\rangle \mid \sum_1^n a_ib_i = 1\}$ and $|P| = 2^{2n-1} - 2^{n-1}$.

Let $\alpha = e_1 + f_1 \in P$. Note that if δ is orthogonal to α and ϵ is not, then δ and ϵ are in different orbits of the action of G_α on P. With no loss, we assume that $\Delta(\alpha)$ consists of elements of P which are not orthogonal to α and $\Phi(\alpha)$ consists of elements in $P \setminus \{\alpha\}$ which are orthogonal to α. We have

$$\begin{aligned}\Delta(\alpha) &= \{\langle v\rangle \in P \mid (e_1 + f_1, v) = 1\}\\ &= \{\langle \textstyle\sum_1^n (a_ie_i + b_if_i)\rangle \in P \mid a_1 + b_1 = 1\}\\ &= \{\langle e_1 + v_1\rangle, \langle f_1 + v_1\rangle \mid \langle v_1\rangle \text{ is a nonsingular point in } V_1\},\end{aligned}$$

where $V_1 = \langle e_2, ..., e_n, f_2, ..., f_n\rangle$. So $a = |\Delta(\alpha)| = 2^{2n-2} - 2^{n-1}$ and therefore, $b = |\Phi(\alpha)| = 2^{2n-2} - 1$.

Let $\beta = \langle e_1 + e_2 + f_2\rangle \in \Delta(\alpha)$. Then

$$\begin{aligned}\Delta(\alpha) \cap \Delta(\beta) &= \{\langle v\rangle \in P \mid (e_1 + f_1, v) = (e_1 + e_2 + f_2, v) = 1\}\\ &= \{\langle \textstyle\sum_1^n (a_ie_i + b_if_i)\rangle \in P \mid a_1 + b_1 = b_1 + a_2 + b_2 = 1\}\end{aligned}.$$

An easy calculation shows that

$$r = |\Delta(\alpha) \cap \Delta(\beta)| = 2^{2n-3} - 2^{n-2}.$$

Similarly, if $\gamma = \langle e_2 + f_2\rangle \in \Phi(\alpha)$ then we have

$$\Delta(\alpha) \cap \Delta(\gamma) = \{\langle v\rangle \in P \mid a_1 + b_1 = a_2 + b_2 = 1\}$$

and

$$s = |\Delta(\alpha) \cap \Delta(\gamma)| = 2^{2n-3} - 2^{n-1}.$$

Now equation (2.1) becomes

$$x^2 + 2^{n-2}x - 2^{2n-3} = 0,$$

which has two roots 2^{n-2} and -2^{n-1}. By Lemma 2.1, $U'_c = S(\mathbb{F}P)$ for any $c \neq 2^{n-2}, -2^{n-1}$. Similarly to the study of permutation modules for finite classical group acting on singular points (see [L1, L2]), the graph submodules $U'_{2^{n-2}}$ and $U'_{-2^{n-1}}$ of $\mathbb{F}P$ are minimal when $\mathrm{char}(\mathbb{F}) \neq 2$ in the following sense.

Proposition 3.1. *Suppose that the characteristic of $\mathbb{F}$ is odd. Then every nonzero $\mathbb{F}G$-submodule of $\mathbb{F}P$ either is $T(\mathbb{F}P)$ or contains a graph submodule, which is $U'_{2^{n-2}}$ or $U'_{-2^{n-1}}$.*

We use some ideas from the proof of a similar result for the permutation module of G acting on singular points (see [L1]) but the proof presented here is much simpler. Define

$$\Delta_1 = \{\langle \sum_{i=1}^{n} (a_i e_i + b_i f_i) \rangle \in P \mid b_1 = 1, a_2 + b_2 = 1\}$$

and

$$\Delta_2 = \{\langle \sum_{i=1}^{n} (a_i e_i + b_i f_i) \rangle \in P \mid b_1 = 1, a_2 + b_2 = 0\}.$$

Putting $\Delta = \Delta_1 \cup \Delta_2$ and $\Phi = P \backslash \Delta$. It is easy to see that

$$[\Delta(\langle e_2 + f_2 \rangle)] - [\Delta(\langle e_1 + e_2 + f_2 \rangle)] = [\Delta_1] - [\Delta_2]$$

and

$$\Phi = \{\langle \sum_{i=1}^{n} (a_i e_i + b_i f_i) \rangle \in P \mid b_1 = 0\}.$$

Consider a subgroup $H < G$ consisting of orthogonal transformations sending elements of the basis $\{e_1, f_1, e_2, f_2, ..., e_n, f_n\}$ to those of the basis $\{e_1, f_1 + \sum_{i=1}^{n} a_i e_i + \sum_{i=2}^{n} b_i f_i, e_2 - b_2 e_1, f_2 - a_2 e_1, ..., e_n - b_n e_1, f_n - a_n e_1\}$ respectively, where $a_i, b_i \in \mathbb{F}_2$ and $a_1 = \sum_{i=2}^{n} a_i b_i$. Let K be the subgroup of H consisting of transformations fixing $e_2 + f_2$. Let P_1 be the set of nonsingular points in $V_1 = \langle e_2, f_2, ..., e_n, f_n \rangle$. For each $\langle w \rangle \in P_1$, we define $B_w = \{\langle w \rangle, \langle e_1 + w \rangle\}$. The following lemmas are similar to Propositions 2.1 and 2.2 of [L1] and pretty straightforward. Therefore we skip their proofs.

Lemma 3.2. *The following holds:*

(i) $|H| = 2^{2n-2}$, $|K| = 2^{2n-3}$, $|\Delta| = 2^{2n-2}$, *and* $|\Delta_1| = |\Delta_2| = 2^{2n-3}$;

(ii) *H acts transitively on Δ and K has 2 orbits Δ_1, Δ_2 on Δ.*

Lemma 3.3. *The following holds:*

(i) $\Phi = \bigcup_{\langle w \rangle \in P_1} B_w$;

(ii) *K fixes $B_{e_2+f_2}$ point-wise and is transitive on B_w for every $\langle e_2 + f_2 \rangle \neq \langle w \rangle \in P_1$;*

(iii) *H acts transitively on B_w for every $\langle w \rangle \in P_1$.*

Proof of Proposition 3.1. Suppose that U is a nonzero submodule of $\mathbb{F}P$. We assume that U is not $T(\mathbb{F}P)$. Then U contains an element of the form

$$u = a\langle\phi_1\rangle + b\langle\phi_2\rangle + \sum_{\delta\in P\setminus\{\langle\phi_1\rangle,\langle\phi_2\rangle\}} a_\delta\delta,$$

where $a, b, a_\delta \in \mathbb{F}$, $\langle\phi_1\rangle \neq \langle\phi_2\rangle \in P$, and $a \neq b$. If $(\phi_1, \phi_2) = 1$, we choose an element $\langle\phi_3\rangle \in P$ so that $(\phi_1, \phi_3) = (\phi_2, \phi_3) = 0$. Since $a \neq b$, the coefficient of $\langle\phi_3\rangle$ in u is different from either a or b. Therefore, with no loss, we can assume $(\phi_1, \phi_2) = 0$.

Since $(e_2+f_2, e_1+e_2+f_2) = 0$, there exists $g' \in G$ such that $\phi_1 g' = e_2+f_2$ and $\phi_2 g' = e_1 + e_2 + f_2$. Therefore, we can assume that $u = a\langle\phi_1\rangle + b\langle\phi_2\rangle + \sum_{\delta\in P\setminus\{\phi_1,\phi_2\}} a_\delta\delta \in U$ with $\phi_1 = e_2 + f_2$ and $\phi_2 = e_1 + e_2 + f_2$.

Take an element $g \in G$ such that $(e_2 + f_2)g = e_1 + e_2 + f_2$. Then $\langle\phi_1\rangle g = \langle\phi_2\rangle$ and $\langle\phi_2\rangle g = \langle\phi_1\rangle$. So we have

$$u - ug = (a - b)\langle\phi_1\rangle - (a - b)\langle\phi_2\rangle + \sum_{\delta\in P\setminus\{\langle\phi_1\rangle,\langle\phi_2\rangle\}} b_\delta\delta \in U,$$

where $b_\delta \in \mathbb{F}$. Note that $u - ug \in S(\mathbb{F}P)$. Therefore, if $c_\delta = b_\delta/(a - b)$, we get

$$u_1 := (u - ug)/(a - b) = \langle\phi_1\rangle - \langle\phi_2\rangle + \sum_{\delta\in P\setminus\{\langle\phi_1\rangle,\langle\phi_2\rangle\}} c_\delta\delta \in U \cap S(\mathbb{F}P).$$

Hence we have $u_2 := \sum_{k\in K} u_1 k \in U \cap S(\mathbb{F}P)$. Moreover, by Lemmas 3.2 and 3.3,

$$u_2 = 2^{2n-3}(\langle\phi_1\rangle - \langle\phi_2\rangle) + \sum_{\delta\in\Delta} d_\delta\delta + \sum_{\langle w\rangle\in P_1, w\neq\phi_1} d_w[B_w],$$

where $d_\delta, d_w \in \mathbb{F}$. Therefore $u_3 := \sum_{h\in H} u_2 h \in U \cap S(\mathbb{F}P)$ with

$$u_3 = (\sum_{\delta\in\Delta} d_\delta)[\Delta] + 2^{2n-2} \sum_{\langle w\rangle\in P_1, w\neq\phi_1} d_w[B_w]$$

again by Lemmas 3.2 and 3.3. It follows that

$$u_4 := 2^{2n-2}u_2 - u_3 = 2^{4n-5}(\langle\phi_1\rangle - \langle\phi_2\rangle) + \sum_{\delta\in\Delta} f_\delta\delta \in U \cap S(\mathbb{F}P),$$

where $f_\delta = 2^{2n-2}d_\delta - \sum_{\delta\in\Delta} d_\delta$. Hence

$$u_5 := \sum_{k\in K} u_4 k = 2^{6n-8}(\langle\phi_1\rangle - \langle\phi_2\rangle) + f[\Delta_1] + f'[\Delta_2] \in U \cap S(\mathbb{F}P),$$

where $f = \sum_{\delta\in\Delta_1} f_\delta$ and $f' = \sum_{\delta\in\Delta_2} f_\delta$. Since $u_5 \in S(\mathbb{F}P)$, $f + f' = 0$. Note that $[\Delta_1] - [\Delta_2] = [\Delta(\langle\phi_1\rangle)] - [\Delta(\langle\phi_2\rangle)]$. Therefore,

$$u_5 = 2^{6n-8}(\langle\phi_1\rangle - \langle\phi_2\rangle) + f([\Delta(\langle\phi_1\rangle)] - [\Delta(\langle\phi_2\rangle)]) \in U. \tag{3.5}$$

Case 1: If $f = 0$ then $u_5 = 2^{6n-8}(\langle\phi_1\rangle - \langle\phi_2\rangle) \in U$. From the hypothesis that the characteristic of $\mathbb{F}$ is odd, we have $2^{6n-8} \neq 0$. It follows that $\langle\phi_1\rangle - \langle\phi_2\rangle \in U$.

Therefore $\alpha-\beta\in U$ for every $\alpha,\beta\in P$. In other words, $U\supseteq S(\mathbb{F}P)$, which implies that U contains both $U'_{-2^{n-1}}$ and $U'_{2^{n-2}}$, as wanted.

Case 2: If $f\neq 0$ then (3.5) implies that $(2^{6n-8}/f)(\langle\phi_1\rangle-\langle\phi_2\rangle)+[\Delta(\langle\phi_1\rangle)]-[\Delta(\langle\phi_2\rangle)]\in U$. Therefore, $(2^{6n-8}/f)(\alpha-\beta)+[\Delta(\alpha)]-[\Delta(\beta)]\in U$ for every $\alpha,\beta\in P$. In other words, $U\supseteq U'_{2^{6n-8}/f}$. This and Lemma 2.1 imply that U contains either $U'_{-2^{n-1}}$ or $U'_{2^{n-2}}$. The proposition is completely proved. □

Proposition 3.4. *If $\ell=\mathrm{char}(\mathbb{F})\neq 2,3$, then*

$$\dim U'_{2^{n-2}}=\frac{(2^n-1)(2^{n-1}-1)}{3}\ \textit{and}\ \dim U'_{-2^{n-1}}=\frac{2^{2n}-4}{3}.$$

Proof. We have $v_{2^{n-2},\alpha}-v_{-2^{n-1},\alpha}=3\cdot 2^{n-2}\alpha$ for any $\alpha\in P$. It follows that, for any $\alpha,\beta\in P$,

$$(v_{2^{n-2},\alpha}-v_{2^{n-2},\beta})-(v_{-2^{n-1},\alpha}-v_{-2^{n-1},\beta})=3\cdot 2^{n-2}(\alpha-\beta).$$

Since $\ell\neq 2,3$, $U'_{2^{n-2}}+U'_{-2^{n-1}}=S(\mathbb{F}P)$. Note that $T(v)=-2^{n-1}v$ for any $v\in U'_{2^{n-2}}$ and $T(v)=2^{n-2}v$ for any $v\in U'_{-2^{n-1}}$. Therefore $U'_{2^{n-2}}\cap U'_{-2^{n-1}}=\{0\}$ since $2^{n-2}\neq -2^{n-1}$. So we have $U'_{2^{n-2}}\oplus U'_{-2^{n-1}}=S(\mathbb{F}P)$. In particular,

$$\dim U'_{2^{n-2}}+\dim U'_{-2^{n-1}}=|P|-1=2^{2n-1}-2^{n-1}-1. \tag{3.6}$$

Let A be the matrix of the linear transformation $T|_{S(\mathbb{F}P)}:S(\mathbb{F}P)\to S(\mathbb{F}P)$ (see (2.2)) corresponding to the basis $\{\alpha-\beta\mid\beta\in P\setminus\alpha\}$ with a fixed $\alpha\in P$. Recall that $T(\alpha-\beta)=[\Delta(\alpha)]-[\Delta(\beta)]$ and $\beta\notin\Delta(\beta)$ for every $\beta\in P$. It is easy to see that the trace of A is $-a=-(2^{2n-2}-2^{n-1})$. Since A has two different eigenvalues -2^{n-2} and 2^{n-1} with corresponding eigenvector spaces $U'_{-2^{n-1}}$ and $U'_{2^{n-2}}$, we have

$$-2^{n-2}\dim U'_{-2^{n-1}}+2^{n-1}\dim U'_{2^{n-2}}=-(2^{2n-2}-2^{n-1}). \tag{3.7}$$

Now (3.6) and (3.7) imply the proposition. □

As we mentioned in the introduction, the structure of $\mathbb{F}P$ is more complicated when $2^{n-2}=-2^{n-1}$ or equivalently when $\ell=3$. We now study the decomposition of $\overline{\rho}$ (when $\ell=3$) into irreducible characters, which are also called *constituents*.

Lemma 3.5. *Suppose $\ell=3$. Then*

(i) *when n is even, $\overline{\rho}$ has exactly 2 trivial constituents, 2 constituents of degree $(2^n-1)(2^{n-1}-1)/3$, and 1 constituent of degree $(2^n-1)(2^{n-1}+2)/3-2$;*
(ii) *when n is odd, $\overline{\rho}$ has exactly 1 trivial constituent, 2 constituents of degree $(2^n-1)(2^{n-1}-1)/3$, and 1 constituent of degree $(2^n-1)(2^{n-1}+2)/3-1$.*

Proof. The case $n=3$ can be checked directly using [Atl1]. Therefore we assume $n\geq 4$. From the proof of Proposition 3.4, if $\ell=0$ we have $\mathbb{F}P=T(\mathbb{F}P)\oplus U'_{2^{n-2}}\oplus U'_{-2^{n-1}}$. Let φ and ψ be irreducible complex characters of G afforded by $U'_{2^{n-2}}$ and $U'_{-2^{n-1}}$, respectively. Then we have $\rho=1+\varphi+\psi$, where $\varphi(1)=(2^n-1)(2^{n-1}-1)/3$ and

$\psi(1) = (2^{2n} - 4)/3$. It has been shown in [Ho] that the smallest degree of non-linear irreducible characters of G in cross-characteristic is at least $(2^n - 1)(2^{n-1} - 1)/3$. Therefore $\overline{\varphi}$ must be irreducible.

It remains to consider $\overline{\psi}$. From [L2], the complex permutation character ρ^0 of G acting on singular points has 3 constituents of degrees 1, $(2^n - 1)(2^{n-1} + 2)/3$, and $(2^{2n} - 4)/3$. It follows that, by Lemma 2.3, ψ must be a common constituent of ρ and ρ^0. The lemma follows by using the decomposition of $\overline{\psi}$ from §6 of [ST]. □

Proof of Theorem 1.1 when $G = O^+_{2n}(2)$. We consider four cases as described in Table 1:

(i) $\ell \neq 2, 3; \ell \nmid (2^n - 1)$: Then we have $[P] = 2^{2n-1} - 2^{n-1} \neq 0$ and therefore $S(\mathbb{F}P) \cap T(\mathbb{F}P) = \{0\}$. So $\mathbb{F}P = S(\mathbb{F}P) \oplus T(\mathbb{F}P)$. From the proof of Proposition 3.4, we see that $U'_{2^{n-2}} \oplus U'_{-2^{n-1}} = S(\mathbb{F}P)$. So

$$\mathbb{F}P = T(\mathbb{F}P) \oplus U'_{2^{n-2}} \oplus U'_{-2^{n-1}}.$$

By Proposition 3.1, these direct summands are simple and their dimensions are given in Proposition 3.4. In Table 1, $X := U'_{2^{n-2}}$ and $Y = U'_{-2^{n-1}}$.

(ii) $\ell \neq 2, 3; \ell \mid (2^n - 1)$: Then we have $T(\mathbb{F}P) \subset S(\mathbb{F}P)$. Since $\ell \neq 3$ and $\ell \mid (2^n - 1)$, $\ell \nmid (2^{n-2} - 1)$ and therefore $2^{2n-2} - 2^{n-1} \neq 2^{n-1}$. Recall that $T(v) = av = (2^{2n-2} - 2^{n-1})v$ for any $v \in T(\mathbb{F}P)$ and $T(v) = 2^{n-1}v$ for any $v \in U'_{2^{n-2}}$. Therefore

$$T(\mathbb{F}P) \cap U'_{2^{n-2}} = \{0\}. \tag{3.8}$$

Hence, by Proposition 3.1, $U'_{2^{n-2}}$ is simple.

If c is a root of the quadratic equation $x^2 + (r - s)x + (s - a) = 0$, then $\sum_{\delta \in \Delta(\alpha)} v_{c,\delta} = (a - s)\alpha + (r - s + c)[\Delta(\alpha)] + s[P] \in U_c$. Therefore,

$$s[P] = (a - s)\alpha + (r - s + c)[\Delta(\alpha)] + s[P] - (r - s + c)(c\alpha + [\Delta(\alpha)]) \in U_c.$$

Recall that $s = 2^{2n-3} - 2^{n-1} = 2^{n-1}(2^{n-2} - 1) \neq 0$ as $\ell \nmid (2^{n-2} - 1)$. It follows that $[P] \in U_{2^{n-2}}$ and $[P] \in U_{-2^{n-1}}$. Equivalently,

$$T(\mathbb{F}P) \in U_{2^{n-2}} \text{ and } T(\mathbb{F}P) \in U_{-2^{n-1}}. \tag{3.9}$$

Now (3.8) and (3.9) imply that $U_{2^{n-2}} = U'_{2^{n-2}} \oplus T(\mathbb{F}P)$. Notice that $v_{2^{n-2},\alpha} - v_{-2^{n-1},\alpha} = 3 \cdot 2^{n-2}\alpha$ for any $\alpha \in P$. So $\mathbb{F}P = U_{2^{n-2}} + U_{-2^{n-1}}$. It follows that $\mathbb{F}P = U'_{2^{n-2}} + U_{-2^{n-1}}$ since $T(\mathbb{F}P) \in U_{-2^{n-1}}$. We also have $\dim U'_{2^{n-2}} + \dim U_{-2^{n-1}} \leq \dim U'_{2^{n-2}} + \dim U'_{-2^{n-1}} + 1 = \dim S(\mathbb{F}P) + 1 = |P|$. So

$$\mathbb{F}P = U'_{2^{n-2}} \oplus U_{-2^{n-1}}.$$

If $T(\mathbb{F}P) \nsubseteq U'_{-2^{n-1}}$ then $U_{-2^{n-1}} = T(\mathbb{F}P) \oplus U'_{-2^{n-1}}$. It follows that $(T(\mathbb{F}P) \oplus U'_{-2^{n-1}}) \cap U'_{2^{n-2}} = 0$, which leads to a contradiction since $T(\mathbb{F}P) \subset S(\mathbb{F}P) = U'_{-2^{n-1}} \oplus$

$U'_{2^{n-2}}$. So we have $T(\mathbb{F}P) \subset U'_{-2^{n-1}}$. We deduce that, by Proposition 3.1, $U_{-2^{n-1}}$ is uniserial with composition series

$$0 \subset T(\mathbb{F}P) \subset U'_{-2^{n-1}} \subset U_{-2^{n-1}}.$$

It is easy to see that $v_{-2^{n-1},\alpha} - v_{-2^{n-1},\alpha}g = v_{-2^{n-1},\alpha} - v_{-2^{n-1},\alpha g} \in U'_{-2^{n-1}}$. Therefore $U_{-2^{n-1}}/U'_{-2^{n-1}}$ is isomorphic to the one-dimensional trivial $\mathbb{F}G$-module. If we put $Y := U'_{-2^{n-1}}/T(\mathbb{F}P)$, then the socle series of $U_{-2^{n-1}}$ is $\mathbb{F} - Y - \mathbb{F}$ (here and after, we use row notation for socle series). We note that, in Table 1, $X := U'_{2^{n-2}}$.

(iii) $\ell = 3, n$ even: Then $2^{n-2} = -2^{n-1} = 1$. Let $S \subset P$ be the set of all nonsingular points of the form $\langle e_1 + f_1 + v\rangle$ with $v \in \langle e_2, e_3, ..., e_n\rangle$. It is easy to check that

$$\sum_{\alpha\in S} v_{1,\alpha} = \sum_{\alpha\in S}(\alpha + [\Delta(\alpha)]) = \sum_{\alpha\in S}\alpha + 2^{n-2}\sum_{\alpha\in P\setminus S}\alpha = [P].$$

Therefore, $[P] \in U_1$. Moreover, since $|S| = 2^{n-1} \neq 0$, $[P] \notin U'_1$. Therefore we have $U_1 = T(\mathbb{F}P) \oplus U'_1$ and U'_1 is simple by Proposition 3.1. Moreover, U_1 is the socle of $\mathbb{F}P$.

By Lemma 2.1, $\langle v_{1,\alpha}, v_{1,\beta}\rangle = s = 2^{2n-3} - 2^{n-1} = 0$ for any $\alpha, \beta \in P$. Hence $\langle U_1, U_1\rangle = 0$ or equivalently, $U_1 \subseteq U_1^\perp$. A similar argument to the proof of Lemma 2.2 of [ST] shows that U_1 self-dual. Therefore, $\mathbb{F}P/U_1^\perp \cong \mathrm{Hom}_{\mathbb{F}}(U_1, \mathbb{F}) \cong U_1$. We have shown that $0 \subset U_1 \subseteq U_1^\perp \subset \mathbb{F}P$ is a series of $\mathbb{F}P$ with $\mathbb{F}P/U_1^\perp \cong U_1 \cong T(\mathbb{F}P) \oplus U'_1$. This and Lemma 3.5 imply that $\dim U'_1 = (2^n - 1)(2^{n-1} - 1)/3$ and $U_1^\perp/U_1$ is simple of dimension $(2^n - 1)(2^{n-1} + 2)/3 - 2$.

Setting $X := U'_1$ and $Z := U_1^\perp/U_1$. Then the composition factors of $\mathbb{F}P$ are: $\mathbb{F}$ (twice), X (twice), and Z. By Proposition 3.1, $S(\mathbb{F}P)/T(\mathbb{F}P)$ is uniserial with socle series $X - Z - X$. Note that ${U'_1}^\perp/U'_1$ has composition factors: $\mathbb{F}$ (twice) and Z. We will show that ${U'_1}^\perp/U'_1$ is also uniserial with socle series $\mathbb{F} - Z - \mathbb{F}$. Assume the contrary. Then Z would be (isomorphic to) a submodule of ${U'_1}^\perp/U'_1$. The self-duality of ${U'_1}^\perp/U'_1$ then implies that Z is a direct summand of ${U'_1}^\perp/U'_1$. By (2.3), $\mathbb{F}P/\operatorname{Ker}(Q) \cong \operatorname{Im}(Q)$, which is neither 0 nor $T(\mathbb{F}P^0)$. Inspecting the structure of $\mathbb{F}P^0$ given in Figure 4 of [ST], we see that $\operatorname{Im}(Q)$ has a uniserial submodule $\mathbb{F} - Z$. It follows that $\mathbb{F}P/\operatorname{Ker}(Q)$ also has a uniserial submodule $\mathbb{F} - Z$, which leads to a contradiction since Z is a direct summand of ${U'_1}^\perp/U'_1$. We conclude that the socle series of $\mathbb{F}P$ is $(\mathbb{F} \oplus X) - Z - (\mathbb{F} \oplus X)$.

(iv) $\ell = 3, n$ odd: Then $2^{n-2} = -2^{n-1} = 2$. Since $|P| = 2^{2n-1} - 2^{n-1} \neq 0$, $[P] \notin S(\mathbb{F}P)$ and therefore $\mathbb{F}P = T(\mathbb{F}P) \oplus S(\mathbb{F}P)$. Similar to (iii), we have $U_2 = T(\mathbb{F}P) \oplus U'_2$ and U'_2 is simple. Moreover, U'_2 is the socle of $S(\mathbb{F}P)$. We also have that U_2 as well as U'_2 are self-dual. By Lemma 2.1, $\langle U'_2, U_2\rangle = 0$, whence $U'_2 \subseteq U_2^\perp$. Since $[P] \in U_2$, $U_2^\perp \subset S(\mathbb{F}P)$ and we obtain $S(\mathbb{F}P)/U_2^\perp \cong \mathbb{F}P/{U'_2}^\perp \cong \mathrm{Hom}_{\mathbb{F}}(U'_2, \mathbb{F}) \cong U'_2$. Combining this with Lemma 3.5, we have $\dim U'_2 = (2^n - 1)(2^{n-1} - 1)/3$ and $U_2^\perp/U'_2$ is simple of dimension $(2^n - 1)(2^{n-1} + 2)/3 - 1$. By Lemma 2.4 and the self-duality of $S(\mathbb{F}P)$,

$S(\mathbb{F}P)/U_2^{\perp}$ is the top layer of $S(\mathbb{F}P)$ and therefore $U_2^{\perp}/U_2'$ is the second layer. Setting $X = U_2'$ and $Z = U_2^{\perp}/U_2'$, then the socle series of $S(\mathbb{F}P)$ is $X - Z - X$. □

4. The orthogonal groups $O^-_{2n}(2)$

Let $Q(\cdot)$ be a quadratic form of type $-$ on a vector space V of dimension $2n \geq 6$ and $(.,.)$ be a symmetric bilinear form associated with Q so that $Q(au + bv) = a^2Q(u) + b^2Q(v) + ab(u, v)$ for any $a, b \in \mathbb{F}_2, u, v \in V$. Then $G = O^-_{2n}(2)$ is the orthogonal group of linear transformations of V preserving Q. We choose a basic of V consisting of vectors $\{e_1, ..., e_n, f_1, ..., f_n\}$ so that $Q(e_i) = Q(f_j) = 0$, $(e_i, f_j) = \delta_{ij}$, $(e_i, e_n) = (e_i, f_n) = (f_j, e_n) = (f_j, f_n) = 0$ for every $i, j = 1, ..., n-1$ and $Q(e_n) = Q(f_n) = (e_n, f_n) = 1$. If P is the set of all nonsingular points in V then $P = \{\langle \sum_1^n (a_ie_i + b_if_i)\rangle \mid a_n + b_n + \sum_1^n a_ib_i = 1\}$ and $|P| = 2^{2n-1} + 2^{n-1}$.

A similar calculation as in §3 shows that $a = 2^{2n-2} + 2^{n-1}, b = 2^{2n-2} - 1, r = 2^{2n-3} + 2^{n-2}, s = 2^{2n-3} + 2^{n-1}$. Equation (2.1) now becomes $x^2 - 2^{n-2}x - 2^{2n-3} = 0$ and it has two roots -2^{n-2} and 2^{n-1}.

Proposition 4.1. *Suppose that the characteristic of $\mathbb{F}$ is odd. Then every nonzero $\mathbb{F}G$-submodule of $\mathbb{F}P$ either is $T(\mathbb{F}P)$ or contains a graph submodule, which is $U'_{-2^{n-2}}$ or $U'_{2^{n-1}}$.*

Proof. Define $\Delta_1, \Delta_2, \Delta, \Phi, P_1, V_1$, and B_w similarly as in §3. Consider a subgroup $H < G$ consisting of orthogonal transformations sending elements of the basis $\{e_1, f_1, e_2, f_2, ..., e_n, f_n\}$ to those of the basis $\{e_1, f_1 + \sum_{i=1}^n a_ie_i + \sum_{i=2}^n b_if_i, e_2 - b_2e_1, f_2 - a_2e_1, ..., e_n - b_ne_1, f_n - a_ne_1\}$ respectively, where $a_i, b_i \in \mathbb{F}_2$ and $a_1 = a_n + b_n + \sum_{i=2}^n a_ib_i$. Let K be the subgroup of H consisting of transformations fixing $e_2 + f_2$. Then Lemmas 3.2 and 3.3 are still true in this case. Now we just argue exactly the same as in the proof of Proposition 3.1. □

Proposition 4.2. *If $\ell = \mathrm{char}(\mathbb{F}) \neq 2, 3$, then*

$$\dim U'_{-2^{n-2}} = \frac{(2^n+1)(2^{n-1}+1)}{3} \text{ and } \dim U'_{2^{n-1}} = \frac{2^{2n}-4}{3}.$$

Proof. Similarly as in Proposition 3.4, we have

$$\dim U'_{-2^{n-2}} + \dim U'_{2^{n-1}} = |P| - 1 = 2^{2n-1} + 2^{n-1} - 1$$

and

$$2^{n-2}\dim U'_{2^{n-1}} - 2^{n-1}\dim U'_{-2^{n-2}} = -(2^{2n-2} + 2^{n-1}).$$

The proposition now follows easily. □

When $\ell = 0$, similarly as in §3, we have

$$\mathbb{F}P = T(\mathbb{F}P) \oplus U'_{-2^{n-2}} \oplus U'_{2^{n-1}}.$$

Let φ and ψ be irreducible complex characters of G afforded by $U'_{-2^{n-2}}$ and $U'_{2^{n-1}}$, respectively. Then we have $\rho = 1 + \varphi + \psi$, where $\varphi(1) = (2^n + 1)(2^{n-1} + 1)/3$ and

$\psi(1) = (2^{2n}-4)/3$. Putting $G_1 := O^+_{2n-2}(2) \leq G$. Note that $G_1 = \Omega^+_{2n-2}(2)\cdot 2$ and $G = \Omega^-_{2n}(2)\cdot 2$. Also, $\Omega^+_{2n-2}(2)$ and $\Omega^-_{2n}(2)$ are simple. Denote by δ and δ_1 the nontrivial 3-Brauer linear characters of G and G_1, respectively. Then we have $\delta|_{G_1} = \delta_1$. The two following lemmas describe the decompositions of restrictions of $\overline{\rho}$, $\overline{\varphi}$, and $\overline{\psi}$ to G_1 when $\ell = 3$.

Lemma 4.3. *Suppose $\ell = 3$ and n is even. Then*

(i) $\overline{\rho}|_{G_1}$ *has exactly* 7 *trivial constituents,* 3 *constituents* δ_1, 5 *constituents of degree* $(2^{n-1}-1)(2^{n-2}-1)/3$, *and* 7 *constituents of degree* $(2^{n-1}-1)(2^{n-2}+2)/3-1$.

(ii) $\overline{\varphi}|_{G_1}$ *has exactly* 5 *constituents of degree* 1, 1 *constituent of degree* $(2^{n-1}-1)(2^{n-2}-1)/3$, *and* 3 *constituents of degree* $(2^{n-1}-1)(2^{n-2}+2)/3-1$.

(iii) $\overline{\psi}|_{G_1}$ *has exactly* 4 *constituents of degree* 1, 4 *constituents of degree* $(2^{n-1}-1)(2^{n-2}-1)/3$, *and* 4 *constituents of degree* $(2^{n-1}-1)(2^{n-2}+2)/3-1$.

Proof. Let P_1 and P_1^0 be the sets of nonsingular and singular points, respectively, in $V_1 = \langle e_1, ..., e_{n-1}, f_1, ..., f_{n-1}\rangle$. We temporarily abuse the notation by identifying a nonzero vector v with the point containing it. Then we have

$$P = \{e_n, f_n, e_n+f_n\} \cup P_1 \cup \{e_n + P_1^0\} \cup \{f_n + P_1^0\} \cup \{e_n + f_n + P_1^0\},$$

where the union is disjoint and by $\{x+Y\}$ we mean $\{x+y \mid y \in Y\}$. Since G_1 fixes e_n, f_n, e_n+f_n and the action of G_1 on each set $\{e_n+P_1^0\}$, $\{f_n+P_1^0\}$, or $\{e_n+f_n+P_1^0\}$ is the same as that of G_1 on P_1^0, we have the following isomorphism as $\mathbb{F}G_1$-modules:

$$\mathbb{F}P \cong 3\mathbb{F} \oplus \mathbb{F}P_1 \oplus 3\mathbb{F}P_1^0.$$

The structure of $\mathbb{F}P_1$ is given in Table 1 and the structure of $\mathbb{F}P_1^0$ is given in Figure 4 of [ST]. Hence part (i) follows.

An easy comparison of degrees using the fact that $\varphi(1) = (2^n+1)(2^{n-1}+1)/3$ and $\psi(1) = (2^{2n}-4)/3$ will prove (ii) and (iii). □

Lemma 4.4. *Suppose $\ell = 3$ and n is odd. Then*

(i) $\overline{\rho}|_{G_1}$ *has exactly* 14 *trivial constituents,* 3 *constituents* δ_1, 5 *constituents of degree* $(2^{n-1}-1)(2^{n-2}-1)/3$, *and* 7 *constituents of degree* $(2^{n-1}-1)(2^{n-2}+2)/3-2$.

(ii) $\overline{\varphi}|_{G_1}$ *has exactly* 8 *constituents of degree* 1, 1 *constituent of degree* $(2^{n-1}-1)(2^{n-2}-1)/3$, *and* 3 *constituents of degree* $(2^{n-1}-1)(2^{n-2}+2)/3-2$.

(iii) $\overline{\psi}|_{G_1}$ *has exactly* 8 *constituents of degree* 1, 4 *constituents of degree* $(2^{n-1}-1)(2^{n-2}-1)/3$, *and* 4 *constituents of degree* $(2^{n-1}-1)(2^{n-2}+2)/3-2$.

Proof. Similar to the proof of Lemma 4.3. □

Lemma 4.5. *Suppose $\ell = 3$. Then $\overline{\psi}$ has exactly* 2 *constituents of degrees* $(2^n+1)(2^{n-1}-2)/3$ *and* $(2^n+1)(2^{n-1}+1)/3-1$ *if n is even and* 3 *constituents of degree* 1, $(2^n+1)(2^{n-1}-2)/3-1$ *and* $(2^n+1)(2^{n-1}+1)/3-1$ *if n is odd.*

Proof. From [L2], we know that the complex permutation character ρ^0 of G acting on singular points has 3 irreducible constituents of degrees 1, $(2^n+1)(2^{n-1}-2)/3$, and $(2^{2n}-4)/3$. Therefore, by Lemma 2.3, ψ must be a common constituent of ρ and ρ^0. Now the lemma follows from §8 of [ST]. □

Proof of Theorem 1.1 when $G = O^-_{2n}(2)$. As described in Table 2, we consider the following cases.

(i) $\ell \neq 2, 3; \ell \nmid (2^n+1)$: Then we have $S(\mathbb{F}P) \cap T(\mathbb{F}P) = \{0\}$ and

$$\mathbb{F}P = T(\mathbb{F}P) \oplus U'_{-2^{n-2}} \oplus U'_{2^{n-1}}.$$

Propositions 4.1 and 4.2 now imply the theorem in this case. In Table 2, $X := U'_{-2^{n-2}}$ and $Y := U'_{2^{n-1}}$.

(ii) $\ell \neq 2, 3; \ell \mid (2^n+1)$: Then we have $T(\mathbb{F}P) \subset S(\mathbb{F}P)$. Since $\ell \neq 3$ and $\ell \mid (2^n+1)$, $\ell \nmid (2^{n-2}+1)$ and therefore $2^{2n-2}+2^{n-1} \neq -2^{n-1}$. Note that $T(v) = av = (2^{2n-2}+2^{n-1})v$ for any $v \in T(\mathbb{F}P)$ and $T(v) = -2^{n-1}v$ for any $v \in U'_{-2^{n-2}}$. It follows that

$$T(\mathbb{F}P) \cap U'_{-2^{n-2}} = \{0\} \tag{4.10}$$

Therefore, by Proposition 4.1, $U'_{-2^{n-2}}$ is simple. Arguing similarly as in the case $G = O^+_{2n}(2)$, we have

$$T(\mathbb{F}P) \in U_{-2^{n-2}} \text{ and } T(\mathbb{F}P) \in U_{2^{n-1}}. \tag{4.11}$$

Now (4.10) and (4.11) imply that $U_{-2^{n-2}} = U'_{-2^{n-2}} \oplus T(\mathbb{F}P)$. Recall that $v_{2^{n-1},\alpha} - v_{-2^{n-2},\alpha} = 3 \cdot 2^{n-2}\alpha$ for any $\alpha \in P$. So $\mathbb{F}P = U_{-2^{n-2}} + U_{2^{n-1}}$. It follows that $\mathbb{F}P = U'_{-2^{n-2}} + U_{2^{n-1}}$ since $T(\mathbb{F}P) \in U_{2^{n-1}}$. We also have $\dim U'_{-2^{n-2}} + \dim U_{2^{n-1}} \leq \dim U'_{-2^{n-2}} + \dim U'_{2^{n-1}} + 1 = \dim S(\mathbb{F}P) + 1 = |P|$. So

$$\mathbb{F}P = U'_{-2^{n-2}} \oplus U_{2^{n-1}}.$$

By Proposition 4.1, $U_{2^{n-1}}$ is uniserial with composition series $0 \subset T(\mathbb{F}P) \subset U'_{2^{n-1}} \subset U_{2^{n-1}}$. Since $v_{2^{n-1},\alpha} - v_{2^{n-1},\alpha}g \in U'_{2^{n-1}}$, $U_{2^{n-1}}/U'_{2^{n-1}}$ is isomorphic to the one-dimensional trivial $\mathbb{F}G$-module. If we put $Y := U'_{2^{n-1}}/T(\mathbb{F}P)$, then $\mathbb{F} - Y - \mathbb{F}$ is the socle series of $U_{2^{n-1}}$. In Table 2, $X := U'_{-2^{n-2}}$.

(iii) $\ell = 3; n$ even: Then we have $-2^{n-2} = 2^{n-1} = 2$. Since $|P| = 2^{2n-1} + 2^{n-1} \neq 0$, $[P] \notin S(\mathbb{F}P)$ and therefore $\mathbb{F}P = T(\mathbb{F}P) \oplus S(\mathbb{F}P)$. As proved before, $s[P] = \sum_{\delta \in \Delta(\alpha)} v_{2,\delta} - (r-s+2)v_{2,\alpha} \in U_2$ for any $\alpha \in P$. Since $|\Delta(\alpha)| - (r-s+2) = (2^{2n-2}+2^{n-1}) - (2^{2n-3}+2^{n-2}-2^{2n-3}-2^{n-1}+2) = 2 \neq 0$, $[P] \notin U'_2$. Applying Proposition 4.1, we see that U'_2 is simple and U'_2 is the socle of $S(\mathbb{F}P)$. By Lemma 2.4, U'_2 is also (isomorphic to) the top layer of the socle series of $S(\mathbb{F}P)$.

By Lemma 2.1, $\langle U'_2, U_2 \rangle = 0$, so $U'_2 \subseteq U_2^{\perp}$. Since $s = 2^{2n-3} + 2^{n-1} \neq 0$, $[P] \in U_2$ and therefore $U_2^{\perp} \in S(\mathbb{F}P)$. Moreover, $S(\mathbb{F}P)/U_2^{\perp} \cong \mathbb{F}P/U_2'^{\perp} \cong \mathrm{Hom}_{\mathbb{F}}(U'_2, \mathbb{F}) \cong U'_2$ by the self-duality of U_2 and $U_2 = T(\mathbb{F}P) \oplus U'_2$. We have shown that U'_2 occurs as a composition factor in $\mathbb{F}P$ with multiplicities at least 2. Let $\sigma \in \mathrm{IBr}_3(G)$ be the irreducible 3-Brauer character of G afforded by U_2'. Then σ is an irreducible constituent of $\overline{\rho} = 1 + \overline{\varphi} + \overline{\psi}$ with multiplicity at leat 2.

Assume that 2σ is contained in $\overline{\varphi}$. Then $\sigma(1) \leq \varphi(1) = (2^n+1)(2^{n-1}+1)/6$, which violates the result on the lower bound of degrees of nontrivial irreducible characters of $\Omega_{2n}^-(2)$ given in [Ho]. So σ is a constituent of $\overline{\psi}$. By Lemma 4.5, $\sigma(1)$ is either $(2^n+1)(2^{n-1}-2)/3$ or $(2^n+1)(2^{n-1}+1)/3-1$ and also σ is a constituent of $\overline{\varphi}$. If $\sigma(1) = (2^n+1)(2^{n-1}-2)/3$, then again from the result on the lower bound of degrees of nontrivial irreducible characters of $\Omega_{2n}^-(2)$, other constituents of $\overline{\varphi}$ are linear. In particular, $\overline{\varphi}$ contains $(2^n+1)(2^{n-1}+1)/3 - (2^n+1)(2^{n-1}-2)/3 = 2^n+1$ linear constituents (counting multiplicities). This contradicts part (i) of Lemma 4.3 since $2^n+1 \geq 17$. So we have $\sigma(1) = (2^n+1)(2^{n-1}+1)/3-1$. Since σ is a constituent of $\overline{\varphi}$ and $\varphi(1) = (2^n+1)(2^{n-1}+1)/3$, $\overline{\varphi}$ has only one another constituent which is linear. Using Lemma 4.3, we see that the restriction of this linear constituent to G_1 is δ_1. Therefore that constituent must be δ.

We have shown that $U_2^\perp/U_2'$ has two composition factors affording δ and a character of degree $(2^n+1)(2^{n-1}-2)/3$. The self duality of $U_2'^\perp/U_2' \cong \mathbb{F} \oplus U_2^\perp/U_2'$ implies that $U_2^\perp/U_2' \cong \delta \oplus Z$, where Z is a module affording the character of degree $(2^n+1)(2^{n-1}-2)/3$. Here we denote by the same δ the module affording character δ. We note that, in Table 2, $X := U_2'$ and the socle series of $S(\mathbb{F}P)$ is $X - (\delta \oplus Z) - X$.

(iv) $\ell = 3, n$ odd: Then we have $-2^{n-2} = 2^{n-1} = 1$. Also, $s = 2^{2n-3} + 2^{n-1} = 0$. Hence, by Lemma 2.1, $\langle U_1, U_1 \rangle = 0$. It follows that $U_1 \subseteq U_1^\perp$. Since U_1 is self-dual, $\mathbb{F}P/U_1^\perp \cong \mathrm{Hom}_{\mathbb{F}}(U_1, \mathbb{F}) \cong U_1$. Therefore, the nontrivial factor of U_1' occurs with multiplicities at least 2 in $\mathbb{F}P$. Note that, by Proposition 4.1, this nontrivial factor is either U_1' or $U_1'/T(\mathbb{F}P)$. Arguing similarly as in (iii) and using Lemmas 4.4 and 4.5, we again obtain that $\overline{\varphi}$ has exactly two irreducible constituents of degrees 1 and $(2^n+1)(2^{n-1}+1)/3-1$. Combining this with Lemma 4.5, we get that $\overline{\rho}$ has exactly 3 linear constituents, 2 constituents of degree $(2^n+1)(2^{n-1}+1)/3-1$, and 1 constituent of degree $(2^n+1)(2^{n-1}-2)/3-1$.

Since $v_{1,\alpha} - v_{1,\alpha}g = v_{1,\alpha} - v_{1,\alpha g} \in U_1'$ for any $\alpha \in P$ and $g \in G$, U_1/U_1' is isomorphic to the one-dimensional trivial module. It follows that, if $T(\mathbb{F}P) \in U_1'$, U_1 would have $\mathbb{F}$ as a composition factor with multiplicities 2. Therefore $\mathbb{F}$ appears at least 4 times as a composition factor in $\mathbb{F}P$, contradicting the previous paragraph. So $T(\mathbb{F}P) \notin U_1'$ and hence U_1' is simple.

Now we have $\dim U_1' = (2^n+1)(2^{n-1}+1)/3-1$. Also, $U_1^\perp/U_1$ has two composition factors of degrees 1 and $(2^n+1)(2^{n-1}-2)/3-1$. Applying Lemma 4.4, the factor of degree 1 must be δ and therefore $U_1^\perp/U_1 \cong Z \oplus \delta$, where Z is the factor of dimension $(2^n+1)(2^{n-1}-2)/3-1$. Setting $X := U_1'$. Then $\mathbb{F}P$ has composition factors: $\mathbb{F}$ (twice), δ, X (twice), and Z. Then, by Proposition 4.1 and Lemma 2.4, $S(\mathbb{F}P)/T(\mathbb{F}T)$ has socle series $X - (Z \oplus \delta) - X$.

Now we study the structure of the self-dual module $U_1'^{\perp}/U_1'$. Note that it has composition factors: $\mathbb{F}$ (twice), δ, and Z. Inspecting the structure of $\mathbb{F}P^0$ given in Figure 7 of [ST], we see that any nontrivial quotient of $\mathbb{F}P^0$ has the uniserial module $Z - \mathbb{F}$ as a quotient. Using Lemma 2.2, we deduce that Z is not a submodule of $U_1'^{\perp}/U_1'$, which implies that Z is in the second layer of the socle series of $U_1'^{\perp}/U_1'$. Again from Lemma 2.2, it is easy to see that δ is a submodule of $U_1'^{\perp}/U_1'$. Therefore, $U_1'^{\perp}/U_1'$ is direct sum of δ with a uniserial module $\mathbb{F} - Z - \mathbb{F}$. The structure of $\mathbb{F}P$ now is determined completely as described. □

5. The unitary groups in even dimensions $U_{2n}(2)$

Let V be a vector space of dimension $2n \geq 4$ over the field of 4 elements $\mathbb{F}_4 = \{0, 1, \tau, \tau^2\}$, where τ is a primitive cubic root of unity. Let $(.,.)$ be a nonsingular conjugate-symmetric sesquilinear form on V, i.e, $(.,.)$ is linear in the first coordinate and $(u, v) = \overline{(v, u)}$ for any $u, v \in V$, where $\overline{x} = x^2$ for $x \in \mathbb{F}_4$. Then $G = U_{2n}(2)$ is the unitary group of linear transformations of V preserving $(.,.)$.

We choose a basic of V consisting of vectors $\{e_1, ..., e_n, f_1, ..., f_n\}$ so that $(e_i, e_j) = (f_i, f_j) = 0$ and $(e_i, f_j) = \delta_{ij}$ for all $i, j = 1, ..., n$. Let P be the set of all nonsingular points in V. Then $P = \{\langle \sum_1^n (a_i e_i + b_i f_i)\rangle \mid \sum_1^n (a_i \overline{b_i} + \overline{a_i} b_i) = 1\}$ and $|P| = (2^{4n-1} - 2^{2n-1})/3$. Unlike the study of permutation modules for orthogonal groups, in this section, for computational convenience we assume that $\Delta(\alpha) \subset P \setminus \{\alpha\}$ consists of elements orthogonal to α and $\Phi(\alpha) \subset P \setminus \{\alpha\}$ consists of elements not orthogonal to α. Then we have $a = (2^{4n-3} + 2^{2n-2})/3, b = 2^{4n-3} - 2^{2n-2} - 1, r = (2^{4n-5} - 2^{2n-3})/3, s = (2^{4n-5} + 2^{2n-2}/3$. Equation (2.1) now becomes $x^2 - 2^{2n-3}x - 2^{4n-5} = 0$ and it has two roots -2^{2n-3} and 2^{2n-2}.

Proposition 5.1. *Suppose that the characteristic of $\mathbb{F}$ is odd. Then every nonzero $\mathbb{F}G$-submodule of $\mathbb{F}P$ either is $T(\mathbb{F}P)$ or contains a graph submodule, which is $U'_{-2^{2n-3}}$ or $U'_{2^{2n-2}}$.*

Proof. We use some ideas from the proof of a similar result for the permutation module of G acting on singular points (see [L1]). Let $\phi_1 := \langle e_2 + \tau f_2\rangle$, $\phi_2 := \langle e_1 + e_2 + \tau f_2\rangle$, $\phi_3 := \langle \tau^2 e_1 + e_2 + \tau f_2\rangle$, and $\phi_4 := \langle \tau e_1 + e_2 + \tau f_2\rangle$. Also $\Delta := \{\langle \sum_{i=1}^n (a_i e_i + b_i f_i)\rangle \in P \mid b_1 = 1\}$, $\Delta_1 := \{\langle \sum_{i=1}^n (a_i e_i + b_i f_i)\rangle \in P \mid b_1 = 1, a_2\tau^2 + b_2 = 0\}$, $\Delta_2 := \{\langle \sum_{i=1}^n (a_i e_i + b_i f_i)\rangle \in P \mid b_1 = 1, a_2\tau^2 + b_2 = 1\}$, $\Delta_3 := \{\langle \sum_{i=1}^n (a_i e_i + b_i f_i)\rangle \in P \mid b_1 = a_2\tau^2 + b_2 = \tau\}$, $\Delta_4 := \{\langle \sum_{i=1}^n (a_i e_i + b_i f_i)\rangle \in P \mid b_1 = 1, a_2\tau^2 + b_2 = \tau^2\}$, and $\Phi := \{\langle \sum_{i=1}^n (a_i e_i + b_i f_i)\rangle \in P \mid b_1 = 0\}$. Then we have

$$\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 \text{ and } P = \Delta \cup \Phi,$$

where the unions are disjoint and

$$[\Delta(\phi_i)] - [\Delta(\phi_j)] = [\Delta_i] - [\Delta_j] \tag{5.12}$$

for any $i, j = 1, 2, 3, 4$.

Consider a subgroup $H < G$ consisting of unitary transformations sending elements of basis $\{e_1, f_1, e_2, f_2, ..., e_n, f_n\}$ to those of basis $\{e_1, f_1 + \sum_{i=1}^n a_i e_i + \sum_{i=2}^n b_i f_i, e_2 - \overline{b_2} e_1, f_2 - \overline{a_2} e_1, ..., e_n - \overline{b_n} e_1, f_n - \overline{a_n} e_1\}$ respectively, where $a_i, b_i \in \mathbb{F}_4$ and $a_1 + \overline{a_1} = \sum_{i=2}^n (\overline{a_i} b_i + a_i \overline{b_i})$. Let K be the subgroup of H consisting of transformations fixing ϕ_1. Let P_1 be the set of nonsingular points in $V_1 = \langle e_2, f_2, ..., e_n, f_n\rangle$. For each $\langle w\rangle \in P_1$, we define $B_{\langle w\rangle} = \{\langle w\rangle, \langle e_1 + w\rangle, \langle \tau e_1 + w\rangle, \langle \tau^2 e_1 + w\rangle\}$. Similarly as in Lemmas 3.2 and 3.3, we have

(i) $|H| = 2^{4n-3}$, $|K| = 2^{4n-5}$, $|\Delta| = 2^{4n-3}$, and $|\Delta_1| = |\Delta_2| = |\Delta_3| = |\Delta_4| = 2^{4n-5}$;
(ii) H acts transitively on Δ and K has 4 orbits $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ on Δ;
(iii) $\Phi = \bigcup_{\langle w\rangle \in P_1} B_{\langle w\rangle}$;
(iv) K fixes B_{ϕ_1} point-wise and is transitive on B_w for every $\phi_1 \neq \langle w\rangle \in P_1$;
(v) H acts transitively on B_w for every $\langle w\rangle \in P_1$.

Suppose that U is a nonzero submodule of $\mathbb{F}P$. Assume that $U \neq T(\mathbb{F}P)$. Similarly as in Proposition 3.1, we can show that U contains an element of the form

$$u = a\phi_1 + b\phi_2 + c\phi_3 + d\phi_4 + \sum_{\delta \in P \setminus \{\phi_1, \phi_2, \phi_3, \phi_4\}} a_\delta \delta,$$

where $a, b, c, d, a_\delta \in \mathbb{F}$ and $a \neq b$. Take an element $g \in G$ such that $(e_2 + \tau f_2)g = e_1 + e_2 + \tau f_2$. Then $\phi_1 g = \phi_2$, $\phi_2 g = \phi_1$, $\phi_3 g = \phi_4$, and $\phi_4 g = \phi_3$. So we have

$$u - ug = (a - b)(\phi_1 - \phi_2) + (c - d)(\phi_3 - \phi_4) + \sum_{\delta \in P \setminus \{\phi_1, \phi_2, \phi_3, \phi_4\}} b_\delta \delta \in U \cap S(\mathbb{F}P),$$

where $b_\delta \in \mathbb{F}$. Now arguing similarly as in Proposition 3.1, we get an element

$$u_5 = (\phi_1 - \phi_2) + e(\phi_3 - \phi_4) + f[\Delta_1] + f'[\Delta_2] + f''[\Delta_3] + f'''[\Delta_4] \in U \cap S(\mathbb{F}P),$$

where $e, f, f', f'', f''' \in \mathbb{F}$. Interchanging ϕ_3 and ϕ_4 if necessary, we suppose that $e \neq -1$.

Let g be an element of H such that $(e_2 + \tau f_2)g = \tau^2 e_1 + e_2 + \tau f_2$. Then $\phi_1 g = \phi_3$, $\phi_3 g = \phi_1$, $\phi_2 g = \phi_4$, and $\phi_4 g = \phi_2$. Also, $\Delta_1 g = \Delta_3$, $\Delta_3 g = \Delta_1$, $\Delta_2 g = \Delta_4$, and $\phi_4 g = \Delta_2$. Therefore,

$$u_6 := u_5 g = (\phi_3 - \phi_4) + e(\phi_1 - \phi_2) + f[\Delta_3] + f'[\Delta_4] + f''[\Delta_1] + f'''[\Delta_2] \in U \cap S(\mathbb{F}P).$$

Note that $f + f' + f'' + f''' = 0$. Therefore, if we set $u_7 := (u_5 + u_6)/(1 + e)$, then

$$u_7 = (\phi_1 - \phi_2 + \phi_3 - \phi_4) + (t + t'')([\Delta_1] - [\Delta_2] + [\Delta_3] - [\Delta_4]) \in U \cap S(\mathbb{F}P),$$

where $t = f/(1 + e)$ and $t'' = f''/(1 + e)$.

Case 1: If $t + t'' \neq 0$ then by (5.12), we have $u_8 := u_7/(t + t'') = v_{c,\phi_1} - v_{c,\phi_2} + v_{c,\phi_3} - v_{c,\phi_4} \in U$, where $c = 1/(t + t'')$. Consider an element $g \in G$ such that $(e_2 + \tau f_2)g = \tau^2(e_1 + e_2 + \tau f_2)$. It is easy to check that $\phi_1 g = \phi_2, \phi_2 g = \phi_3, \phi_3 g = \phi_1$, and $\phi_4 g = \phi_4$. We deduce that $u_9 := u_8 g = v_{c,\phi_2} - v_{c,\phi_3} + v_{c,\phi_1} - v_{c,\phi_4} \in U$. It follows that $u_{10} := u_8 + u_9 = 2(v_{c,\phi_1} - v_{c,\phi_4}) \in U$. Hence $U'_c \subseteq U$, which implies that U contains either $U'_{-2^{2n-3}}$ or $U'_{2^{2n-2}}$ by Lemma 2.1.

Case 2: If $t + t'' = 0$ then $u_7 = \phi_1 - \phi_2 + \phi_3 - \phi_4 \in U$. Let $g \in G$ be the same element as in Case 1. We have $u_7 g = \phi_2 - \phi_3 + \phi_1 - \phi_4 \in U$. It follows that $u_9 := u_7 + u_8 = 2(\phi_1 - \phi_4) \in U$. Therefore $\alpha - \beta \in U$ for every $\alpha, \beta \in P$. In other words, $U \supseteq S(\mathbb{F}P)$, which implies that U contains both $U'_{-2^{2n-3}}$ and $U'_{2^{2n-2}}$, as wanted. □

Proposition 5.2. *If $\ell = \mathrm{char}(\mathbb{F}) \neq 2, 3$, then*

$$\dim U'_{-2^{2n-3}} = \frac{(2^{2n}-1)(2^{2n-1}+1)}{9} \ \textit{and} \ \dim U'_{2^{2n-2}} = \frac{(2^{2n}+2)(2^{2n}-4)}{9}.$$

Proof. Similar to Proposition 3.4, we have

$$\dim U'_{-2^{2n-3}} + \dim U'_{2^{2n-2}} = |P| - 1 = \frac{2^{4n-1} - 2^{2n-1}}{3} - 1$$

and

$$2^{2n-3} \dim U'_{2^{2n-2}} - 2^{2n-2} \dim U'_{-2^{2n-3}} = -a = -\frac{2^{4n-3} + 2^{2n-2}}{3},$$

which imply the proposition. □

In the complex case $\ell = 0$, as before, we have

$$\mathbb{F}P = T(\mathbb{F}P) \oplus U'_{-2^{2n-3}} \oplus U'_{2^{2n-2}} \text{ and } \rho = 1 + \varphi + \psi,$$

where φ and ψ are irreducible complex characters of G afforded by $U'_{-2^{2n-3}}$ and $U'_{2^{2n-2}}$, respectively. Note that $\varphi(1) = (2^{2n}-1)(2^{2n-1}+1)/9$, $\psi(1) = (2^{2n}+2)(2^{2n}-4)/9$.

We now study the decompositions of $\overline{\varphi}$ and $\overline{\psi}$ into irreducible Brauer characters when $\ell = 3$. We note that, when $\ell = 3$, $-2^{2n-3} = 2^{2n-2} = 1$ and therefore $\mathbb{F}P$ has only one graph submodule U'_1.

Lemma 5.3. *Suppose $\ell = 3$. Then $\overline{\psi}$ has exactly 2 constituents of degrees $(2^{2n} - 1)(2^{2n-1} - 2)/9$ and $(2^{2n} - 1)(2^{2n-1} + 1)/9 - 1$ if $3 \nmid n$ and 3 constituents of degrees 1, $(2^{2n} - 1)(2^{2n-1} - 2)/9$, and $(2^{2n} - 1)(2^{2n-1} + 1)/9 - 2$ if $3 \mid n$.*

Proof. From [L1], we know that the complex permutation character ρ^0 of G acting on singular points has 3 constituents of degrees 1, $(2^{2n} - 1)(2^{2n-1} + 4)/3$, and $(2^{2n} + 2)(2^{2n} - 4)/9$. Therefore, by Lemma 2.3, ψ must be a common constituent of ρ and ρ^0. Now the lemma follows from §4 of [ST]. □

Lemma 5.4. *Suppose $\ell = 3$. Then*

(i) *$\overline{\varphi}$ has exactly 2 constituents of degrees $(2^{2n}-1)(2^{2n-1}-2)/9$ and $(2^{2n}-1)/3$.*
(ii) *U'_1 is simple and $\dim U'_1 = (2^{2n} - 1)(2^{2n-1} - 2)/9$.*

Proof. Let S be the set of nonsingular points of the form $\langle e_1+\tau f_1+v\rangle$ where $v\in\langle e_2,e_3,...,e_n\rangle$. It is easy to see that

$$\sum_{\alpha\in S} v_{1,\alpha}=\sum_{\alpha\in S}(\alpha+[\Delta(\alpha)])=[P].$$

Therefore, $T(\mathbb{F}P)\subset U_1$. Moreover, $|S|=4^{n-1}\neq 0$, whence $T(\mathbb{F}P)\nsubseteq U_1'$. By Proposition 5.1, $U_1=U_1'\oplus T(\mathbb{F}P)$ and U_1' is simple, proving the first part of (ii). Since U_1 as well as U_1' are self-dual, $\mathbb{F}P/U_1'^{\perp}\cong \mathrm{Hom}_{\mathbb{F}}(U_1',\mathbb{F})\cong U_1'$. Using the fact $U_1'\subseteq U_1'^{\perp}$ from Lemma 2.1, we have that U_1' occurs at least twice as a composition factor of $\mathbb{F}P$.

For the rest of the proof, we use the method and notation of §4 of [ST]. Let $W':=\langle e_1,...,e_n\rangle$, $W:=\langle f_1,...,f_n\rangle$, $\widetilde{P}:=\mathrm{Stab}_G(W')$, $P:=\widetilde{P}\cap SU_{2n}(2)$, and $Q:=O_2(P)$. Note that Q acts transitively on the set of 2^{2n-1} nonsingular points of the form $\langle f+u\rangle$ where $f\in W$ and $u\in W'$. Let $Q_{\langle f\rangle}:=\mathrm{Stab}_Q(\langle f\rangle)$, then we have

$$\rho|_Q=\sum_{\langle f\rangle\subseteq W}\mathrm{Ind}_{Q_{\langle f\rangle}}^{Q}(1_{Q_{\langle f\rangle}}).$$

Now arguing exactly the same as in [ST], we get the decomposition of $\rho|_{\widetilde{P}}$ into irreducible characters:

$$\rho|_{\widetilde{P}}=1_{\widetilde{P}}+\tau+\zeta+\sigma_0+\sigma_1,$$

where $\tau(1)=(2^{2n}-1)/3-1$, $\zeta(1)=(2^{2n}-1)/3$, $\sigma_0(1)=(2^{2n}-1)(2^{2n-1}-2)/9$, and $\sigma_1(1)=2(2^{2n}-1)(2^{2n-1}-2)/9$. By comparing degrees, we have

$$\varphi|_{\widetilde{P}}=\zeta+\sigma_0 \text{ and } \psi|_{\widetilde{P}}=\tau+\sigma_1.$$

It has been shown also in [ST] that $\overline{\zeta}$ and $\overline{\sigma_0}$ are irreducible. It follows that $\overline{\varphi}$ has at most two constituents. Now we will show that $\overline{\varphi}$ has exactly two constituents. Assume not, then $\overline{\varphi}$ is irreducible. Combining this with Lemma 5.3, we see that $\overline{\rho}$ has no constituent with multiplicity ≥ 2, which contradicts to the fact that U_1' occurs as a composition factor of $\mathbb{F}P$ at least twice.

We have shown that $\overline{\varphi}$ has exactly two constituents of degrees $\zeta(1)=(2^{2n}-1)/3$ and $\sigma_0(1)=(2^{2n}-1)(2^{2n-1}-2)/9$. This shows that $\dim U_1'=(2^{2n}-1)(2^{2n-1}-2)/9$ and the lemma is proved. □

Proof of Theorem 1.1 when $G=U_{2n}(2)$. As described in Table 3, we consider the following cases.

(i) $\ell\neq 2,3;\ell\nmid(2^{2n}-1)$: Then we have $|P|=(2^{4n-1}-2^{2n-1})/3\neq 0$ and hence $S(\mathbb{F}P)\cap T(\mathbb{F}P)=\{0\}$. We then obtain $\mathbb{F}P=T(\mathbb{F}P)\oplus U'_{-2^{2n-3}}\oplus U'_{2^{2n-2}}$. Propositions 5.1 and 5.2 now imply the theorem in this case. In Table 3, $X:=U'_{-2^{2n-3}}$ and $Y:=U'_{2^{2n-2}}$.

(ii) $\ell\neq 2,3;\ell\mid(2^{2n}-1)$: Then $T(\mathbb{F}P)\subset S(\mathbb{F}P)$. Similarly as before, we have $T(\mathbb{F}P)\cap U'_{-2^{2n-3}}=\{0\}$ and therefore, by Proposition 5.1, $U'_{-2^{2n-3}}$ is simple. We also have $\mathbb{F}P = U'_{-2^{2n-3}} \oplus U_{2^{2n-2}}$. Moreover, $U_{2^{2n-2}}$ is uniserial with composition series $0 \subset T(\mathbb{F}P) \subset U'_{2^{2n-2}} \subset U_{2^{2n-2}}$ and socle series $\mathbb{F} - Y - \mathbb{F}$, where $Y := U'_{2^{2n-2}}/T(\mathbb{F}P)$. We note that, in Table 3, $X := U'_{-2^{2n-3}}$.

(iii) $\ell = 3; 3 \mid n$: Then $s = (2^{4n-5} + 2^{2n-2})/3 = 0$. Therefore, by Lemma 2.1, $\langle U_1, U_1 \rangle = 0$, whence $U_1 \subseteq U_1^{\perp}$. From the proof of Lemma 5.4, we know that $U_1 = T(\mathbb{F}P) \oplus U_1'$. Hence, by Lemmas 5.3 and 5.4, $U_1^{\perp}/U_1$ has two composition factors of degrees $(2^{2n}-1)/3$ and $(2^{2n}-1)(2^{2n-1}+1)/9 - 2$, which we denote by W_1 and W_2, respectively. The self-duality of $U_1^{\perp}/U_1$ implies that $U_1^{\perp}/U_1 \cong W_1 \oplus W_2$. Putting $Z := U_1'$. By Proposition 5.1, $S(\mathbb{F}P)/T(\mathbb{F}P)$ is uniserial with socle series $Z - (W_1 \oplus W_2) - Z$. Arguing similarly as in (iv) in the proof of the main theorem for the case §4, we can show that $U_1'^{\perp}/U_1'$ is direct sum of W_1 and a uniserial module $\mathbb{F} - W_2 - \mathbb{F}$. The structure of $\mathbb{F}P$ now is determined as described.

(iv) $\ell = 3, 3 \nmid n$: Then $|P| = (2^{4n-1} - 2^{2n-1})/3 \neq 0$. Therefore, $\mathbb{F}P = T(\mathbb{F}P) \oplus S(\mathbb{F}P)$. Again by Lemma 2.1, $\langle U_1', U_1 \rangle = 0$. Consider a series of $S(\mathbb{F}P)$:

$$0 \subset U_1' \subseteq U_1^{\perp} \subseteq S(\mathbb{F}P).$$

We have $S(\mathbb{F}P)/U_1^{\perp} \cong \mathbb{F}P/U_1'^{\perp} \cong U_1'$. Therefore, by Lemmas 5.3 and 5.4, $U_1^{\perp}/U_1'$ has two composition factors of degrees $(2^{2n}-1)/3$ and $(2^{2n}-1)(2^{2n-1}+1)/9 - 1$, which again we denote by W_1 and W_2, respectively. It is easy to see that $U_1^{\perp}/U_1'$ is self-dual from the self-duality of $U_1'^{\perp}/U_1'$. Therefore we obtain $U_1^{\perp}/U_1' \cong W_1 \oplus W_2$. Putting $Z := U_1'$, then the socle series of $S(\mathbb{F}P)$ is $Z - (W_1 \oplus W_2) - Z$. $\square$

6. The unitary groups in odd dimensions $U_{2n+1}(2)$

Let V be a vector space of dimension $2n+1 \geq 5$ over the field of 4 elements $\mathbb{F}_4 = \{0, 1, \tau, \tau^2\}$, where τ is a primitive cubic root of unity. Let $(.,.)$ be a nonsingular conjugate-symmetric sesquilinear form on V. Then $G = U_{2n+1}(2)$ is the unitary group of linear transformations of V preserving $(.,.)$.

We choose a basic of V consisting of vectors $\{e_1, ..., e_n, f_1, ..., f_n, g\}$ so that $(e_i, e_j) = (f_i, f_j) = (e_i, g) = (f_i, g) = 0$, $(e_i, f_j) = \delta_{ij}$, and $(g, g) = 1$ for all $i, j = 1, ..., n$. Let P be the set of nonsingular points in V. Then $P = \{\langle cg + \sum_1^n (a_i e_i + b_i f_i)\rangle \mid c\bar{c} + \sum_1^n (a_i \overline{b_i} + \overline{a_i} b_i) \neq 0\}$ and $|P| = (2^{4n+1} + 2^{2n})/3$. We assume that $\Delta(\alpha) \subset P \setminus \{\alpha\}$ consists of elements orthogonal to α and $\Phi(\alpha) \subset P \setminus \{\alpha\}$ consists of elements not orthogonal to α. Then we have $a = (2^{4n-1} - 2^{2n-1})/3, b = 2^{4n-1} + 2^{2n-1} - 1, r = (2^{4n-3} + 2^{2n-2})/3, s = (2^{4n-3} - 2^{2n-1})/3$. Equation (2.1) now becomes $x^2 + 2^{2n-2}x - 2^{4n-3} = 0$ and it has two roots 2^{2n-2} and -2^{2n-1}.

Proposition 6.1. *Suppose that the characteristic of $\mathbb{F}$ is odd. Then every nonzero $\mathbb{F}G$-submodule of $\mathbb{F}P$ either is $T(\mathbb{F}P)$ or contains a graph submodule, which is $U'_{2^{2n-2}}$ or $U'_{-2^{2n-1}}$.*

Proof. Define $\phi_1, \phi_2, \phi_3, \phi_4,\ \Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta, \Phi, P_1, V_1$, and $B_{\langle w\rangle}$ similarly as in the proof of Proposition 5.1. Consider a subgroup $H < G$ consisting of unitary transformations sending elements of the basis $\{e_1,\ f_1,\ e_2,\ f_2, ..., e_n, f_n, g\}$ to those of the basis $\{e_1, f_1 + \sum_{i=1}^n a_i e_i + \sum_{i=2}^n b_i f_i + cg, e_2 - \overline{b_2} e_1, f_2 - \overline{a_2} e_1, ..., e_n - \overline{b_n} e_1, f_n - \overline{a_n} e_1, g - \overline{c} e_1\}$ respectively, where $a_i, b_i, c \in \mathbb{F}_4$ and $a_1 + \overline{a_1} + c\overline{c} = \sum_{i=2}^n (\overline{a_i} b_i + a_i \overline{b_i})$. Let K be the subgroup of H consisting of transformations fixing ϕ_1. Now we just argue exactly the same as in the proof of Proposition 5.1. □

Proposition 6.2. *If $\ell = \mathrm{char}(\mathbb{F}) \neq 2, 3$, then*

$$\dim U'_{2^{2n-2}} = \frac{(2^{2n+1}+1)(2^{2n}-1)}{9} \quad \textit{and} \quad \dim U'_{-2^{2n-1}} = \frac{(2^{2n+1}-2)(2^{2n+1}+4)}{9}.$$

Proof. Similar to Proposition 3.4, we have

$$\dim U'_{2^{2n-2}} + \dim U'_{-2^{2n-1}} = |P| - 1 = \frac{2^{4n+1}+2^{2n}}{3} - 1$$

and

$$2^{2n-1} \dim U'_{2^{2n-2}} - 2^{2n-2} \dim U'_{-2^{2n-1}} = -a = -\frac{2^{4n-1}-2^{2n-1}}{3},$$

which imply the proposition. □

In the complex case $\ell = 0$, we have

$$\mathbb{F}P = T(\mathbb{F}P) \oplus U'_{2^{2n-2}} \oplus U'_{-2^{2n-1}} \text{ and } \rho = 1 + \varphi + \psi,$$

where φ and ψ are irreducible complex characters of G afforded by $U'_{2^{2n-2}}$ and $U'_{-2^{2n-1}}$, respectively. Then we have $\varphi(1) = (2^{2n+1}+1)(2^{2n}-1)/9$, $\psi(1) = (2^{2n+1}-2)(2^{2n+1}+4)/9$.

Lemma 6.3. *Suppose $\ell = 3$. Then*

(i) *when $3 \nmid n$, $\overline{\psi}$ has exactly 1 trivial constituent, 2 equal constituents of degree $(2^{2n+1}-2)/3$, 1 constituent of degree $(2^{2n+1}+1)(2^{2n}-4)/9$, and 1 constituent of degree $(2^{2n+1}+1)(2^{2n}-1)/9$;*

(ii) *when $3 \mid n$, $\overline{\psi}$ has exactly 2 trivial constituent, 2 equal constituents of degree $(2^{2n+1}-2)/3$, 1 constituent of degree $(2^{2n+1}+1)(2^{2n}-4)/9-1$, and 1 constituent of degree $(2^{2n+1}+1)(2^{2n}-1)/9$.*

Proof. From [L2], we know that the complex permutation character ρ^0 of G acting on singular points has 3 constituents of degrees 1, $(2^{2n+1}+1)(2^{2n}-4)/3$, and $(2^{2n+1}-2)(2^{2n+1}+4)/9$. Therefore, by Lemma 2.3, ψ must be a common constituent of ρ and ρ^0. Now the lemma follows from §5 of [ST]. □

We will show that $\overline{\varphi}$ is irreducible in any case. In order to do that, we need to recall some results about *Weil characters* of unitary groups. The Weil representation of $SU_m(q)$ with $m \geq 3$ and q a prime power is of degree q^m. Its afforded character decomposes into a sum of $q+1$ irreducibles ones where one of them has degree

$(q^m+(-1)^m q)/(q+1)$ and others have degree $(q^m-(-1)^m)/(q+1)$. These irreducible characters are called (complex) Weil characters of $SU_m(q)$. Each Weil character of $SU_m(q)$ extends to $q+1$ distinct Weil characters of $U_m(q)$ (see Lemma 4.7 of [TZ]). All nonlinear constituents of the reduction modulo ℓ of a complex Weil character are called (ℓ-Brauer) Weil characters. It is well-known that, when $\ell \nmid q$, any Weil character lifts to characteristic 0 and any irreducible Brauer character of degree either $(q^m + (-1)^m q)/(q + 1)$ or $(q^m - (-1)^m)/(q + 1)$ is a Weil character (for instance, see [HM]). The following lemma is a consequence of Lemma 4.2 of [TZ].

Lemma 6.4. *Suppose that ℓ is odd. Then the restriction of a ℓ-Brauer Weil character of $U_{2n+1}(2)$ of degree $(2^{2n+1}-2)/3$ to $U_{2n}(2)$ is a sum of two Weil characters of degree $(2^{2n}-1)/3$.*

We note that, when $\ell = 3$, $2^{2n-2} = -2^{2n-1} = 1$ and therefore $\mathbb{F}P$ has only one graph submodule U_1'.

Lemma 6.5. *Suppose $\ell = 3$. Then*

(i) *U_1' is simple and it occurs as a composition factor of $\mathbb{F}P$ with multiplicity 2.*
(ii) *$\overline{\varphi}$ is irreducible and U_1' affords the character $\overline{\varphi}$. In particular, $\overline{\varphi}(1) = \dim U_1' = (2^{2n+1}+1)(2^{2n}-1)/9$.*

Proof. Let S be the set of nonsingular points in V of the form $\langle g + v\rangle$ where $v \in \langle e_1, ..., e_n\rangle$. It is easy to see that

$$\sum_{\alpha\in S} v_{1,\alpha} = \sum_{\alpha\in S}(\alpha + [\Delta(\alpha)]) = [P].$$

Therefore, $T(\mathbb{F}P) \subset U_1$. Moreover, $|S| = 4^n \neq 0$. It follows that $T(\mathbb{F}P) \not\subseteq U_1'$. By Proposition 6.1, U_1' is simple and $U_1 = U_1' \oplus T(\mathbb{F}P)$, proving the first part of (i). By Lemma 2.1, $\langle U_1', U_1'\rangle = 0$ and therefore $U_1' \subseteq {U_1'}^{\perp}$. Since U_1 as well as U_1' are self-dual, $\mathbb{F}P/{U_1'}^{\perp} \cong \mathrm{Hom}_{\mathbb{F}}(U_1', \mathbb{F}) \cong U_1'$. Hence U_1' occurs at least twice as a composition factor of $\mathbb{F}P$.

Let P_1 and P_1^0 be the sets of nonsingular and singular points, respectively, in $V_1 = \langle e_1, ..., e_n, f_1, ..., f_n\rangle$. Since any nonsingular point in V is either $\langle g\rangle$, a nonsingular point in V_1, or a point of the form $\langle g + v\rangle$ where v is a nonzero singular vector in V_1, we have the following isomorphism of $\mathbb{F}U_{2n}(2)$-modules:

$$\mathbb{F}P \cong \mathbb{F} \oplus \mathbb{F}P_1 \oplus 3\mathbb{F}P_1^0.$$

Combining this isomorphism with the structures of $\mathbb{F}P_1$ given in Table 3 and $\mathbb{F}P_1^0$ given in Figure 1 of [ST], we see that $\overline{\rho}|_{U_{2n}(2)}$ has exactly 4 constituents of degree $(2^{2n}-1)/3$ and at most 15 trivial constituents ($\mathbb{F}P_1$ has 1 constituent of degree $(2^{2n}-1)/3$ and at most 2 trivial constituents; $\mathbb{F}P_1^0$ has 1 constituent of degree $(2^{2n}-1)/3$ and at most 4 trivial constituents). We notice that the constituents of degree $(2^{2n}-1)/3$ are Weil characters of $U_{2n}(2)$. By Lemma 6.3, $\overline{\psi}$ has 2 constituents of degree $(2^{2n+1}-2)/3$.

Again these constituents are Weil characters of G and therefore their restrictions to $U_{2n}(2)$ give us 4 Weil characters of degree $(2^{2n}-1)/3$ by Lemma 6.4. We have shown that $\overline{\varphi}$ has no constituent which is a Weil character.

We will prove (ii) by contradiction. Assume that $\overline{\varphi}$ is reducible. By Lemma 6.3 and the fact that U_1' occurs as a composition factor of $\mathbb{F}P$ at least twice, $\overline{\varphi}$ must have a constituent of degree $(2^{2n+1}+1)(2^{2n}-4)/9$ when $3 \nmid n$ or degree $(2^{2n+1}+1)(2^{2n}-4)/9-1$ when $3 \mid n$. Therefore all other constituents of $\overline{\varphi}$ are of degree $\leq (2^{2n+1}+1)/3+1$ since $\varphi(1)=(2^{2n+1}+1)(2^{2n}-1)/9$. Using Theorem 2.7 of [GMST], we deduce that these constituents are either linear or Weil characters. The latter case does not happen from the previous paragraph. So $\overline{\varphi}$ has at least $(2^{2n+1}+1)/3$ linear constituents, contradicting to the fact that $\overline{\rho}|_{U_{2n}(2)}$ has at most 15 trivial constituents, provided that $n \geq 3$. The case $n=2$ can be handled easily by using [Atl2]. □

Proof of Theorem 1.1 when $G = U_{2n+1}(2)$. As described in Table 4, we consider the following cases.

(i) $\ell \neq 2,3; \ell \nmid (2^{2n+1}+1)$: Then we have $|P| = (2^{4n+1}+2^{2n})/3 \neq 0$ and therefore $S(\mathbb{F}P)\cap T(\mathbb{F}P)=\{0\}$. As before, we have $\mathbb{F}P = T(\mathbb{F}P)\oplus U'_{2^{2n-2}} \oplus U'_{-2^{2n-1}}$, which imply the theorem in this case by Propositions 6.1 and 6.2. In Table 4, $X := U'_{2^{2n-2}}$ and $Y := U'_{-2^{2n-1}}$.

(ii) $\ell \neq 2,3; \ell \mid (2^{2n+1}+1)$: Then $T(\mathbb{F}P)\subset S(\mathbb{F}P)$. Similarly as before, we have $T(\mathbb{F}P)\cap U'_{2^{2n-2}} = \{0\}$ and therefore, by Proposition 6.1, $U'_{2^{2n-2}}$ is simple. We also have $\mathbb{F}P = U'_{2^{2n-2}} \oplus U_{-2^{2n-1}}$. Moreover, $U_{-2^{2n-1}}$ is uniserial with composition series $0 \subset T(\mathbb{F}P) \subset U'_{-2^{2n-1}} \subset U_{-2^{2n-1}}$ and socle series $\mathbb{F}-Y-\mathbb{F}$, where $Y := U'_{-2^{2n-1}}/T(\mathbb{F}P)$. In Table 4, $X := U'_{2^{2n-2}}$.

(iii) $\ell = 3; 3 \mid n$: In the context of Lemmas 6.3 and 6.5, here and after, we will denote by $\mathbb{F}, Z, X$ the simple $\mathbb{F}G$-modules affording the 3-Brauer characters of degrees 1, $(2^{2n+1}-2)/3$, $(2^{2n+1}+1)(2^{2n}-1)/9$, respectively. Also, let W be the irreducible $\mathbb{F}G$-module affording the character of degrees $(2^{2n+1}+1)(2^{2n}-4)/9-1$ when $3 \mid n$ and $(2^{2n+1}+1)(2^{2n}-4)/9$ when $3 \nmid n$. Note that, by Lemma 6.5, we may choose $X = U_1'$.

Also from Lemmas 6.3 and 6.5, when $3 \mid n$, $\mathbb{F}P$ has composition factors: $\mathbb{F}$ (3 times), Z (twice), X (twice), and W. Moreover, $|P| = (2^{4n+1}+2^{2n})/3 \neq 0$ and therefore $\mathbb{F}P = T(\mathbb{F}P)\oplus S(\mathbb{F}P)$. It follows that, by Proposition 6.1, $X = U_1'$ is the socle of $S(\mathbb{F}P)$.

By (2.4), we have $\mathrm{Im}(R|_{S(\mathbb{F}P^0)}) \cong S(\mathbb{F}P^0)/\mathrm{Ker}(R|_{S(\mathbb{F}P^0)})$. Since any nonzero submodule of $S(\mathbb{F}P)$ has socle X, the quotient $S(\mathbb{F}P^0)/\mathrm{Ker}(R|_{S(\mathbb{F}P^0)})$ also has socle X. Inspecting the structure of $S(\mathbb{F}P^0)$ from Figure 2 of [ST], we see that the only quotient of $S(\mathbb{F}P^0)$ having X as the socle is $X-Z-\mathbb{F}-W$. So, $\mathrm{Im}(R|_{S(\mathbb{F}P^0)})$ is uniserial with socle series $X-Z-\mathbb{F}-W$. It follows that the self-dual module $U_1^{\perp}/U_1'$ has a submodule $Z-\mathbb{F}-W$. Notice that $U_1^{\perp}/U_1'$ has composition factor: Z (twice), $\mathbb{F}$

(twice), and W, it must have the structure $Z - \mathbb{F} - W - \mathbb{F} - Z$. We now can conclude that the structure of $S(\mathbb{F}P)$ is $X - Z - \mathbb{F} - W - \mathbb{F} - Z - X$ by Proposition 6.1.

(iv) $\ell = 3; n \equiv 1 \pmod 3$: Then we have $s = (2^{4n-3} - 2^{2n-1})/3 = 0$. Therefore by Lemma 2.1, $\langle U_1, U_1 \rangle = 0$ or equivalently $U_1 \subseteq U_1^{\perp}$. In the proof of Lemma 6.5, we know that $U_1 = T(\mathbb{F}P) \oplus U_1' \cong \mathbb{F} \oplus X$. Also, by the self-duality of U_1, $\mathbb{F}P/U_1^{\perp} \cong U_1$. It follows that, by Lemmas 6.3 and 6.5, $U_1^{\perp}/U_1$ has composition factors: Z (twice) and W.

We will show that $U_1^{\perp}/U_1$ is actually uniserial with socle series $Z - W - Z$. Suppose not, then W would be a submodule of $U_1^{\perp}/U_1$. The self-duality of $U_1^{\perp}/U_1$ then implies that W is a direct summand of $U_1^{\perp}/U_1$. By Proposition 6.1, any nonzero submodule of $\mathbb{F}P$ which is not $T(\mathbb{F}P)$ has $X = U_1'$ in the socle. Therefore, by (2.4), $\mathbb{F}P^0/\operatorname{Ker} R \cong \operatorname{Im}(R)$ has the socle containing X. Inspecting the submodule lattice of $\mathbb{F}P^0$ given in Figure 2 of [ST], we conclude that $\operatorname{Im}(R)$ has the socle series $X - Z - (\mathbb{F} \oplus W)$. This shows that W cannot be a direct summand in $U_1^{\perp}/U_1$, a contradiction.

We have shown that $U_1^{\perp}/U_1$ is uniserial with socle series $Z - W - Z$. Now we temporarily set $\mathbb{F}_1 := T(\mathbb{F}P)$ and $\mathbb{F}_2 := \mathbb{F}P/S(\mathbb{F}P) \cong U_1'^{\perp}/U_1^{\perp}$. From the previous paragraph, $\mathbb{F}P$ has a submodule $\operatorname{Im}(R)$ with socle series $U_1' - Z - (\mathbb{F} \oplus W)$ and the fact that $U_1 = \mathbb{F}_1 \oplus U_1'$, $\operatorname{Im}(R)$ must be $X - Z - (\mathbb{F}_2 \oplus W)$. Arguing similarly, $\mathbb{F}P$ has a quotient having the socle series $(\mathbb{F}_1 \oplus W) - Z - X$. The structure of $\mathbb{F}P$ is now determined as given in Table 4.

(v) $\ell = 3; n \equiv 2 \pmod 3$: Then $|P| = (2^{4n+1} + 2^{2n})/3 \neq 0$ and therefore $\mathbb{F}P = T(\mathbb{F}P) \oplus S(\mathbb{F}P)$. It follows that $S(\mathbb{F}P)$ has composition factors: $\mathbb{F}$, X (twice), Z (twice), and W by Lemmas 6.3 and 6.5.

Proposition 6.1 implies that X is the socle of $S(\mathbb{F}P)$. By Lemma 2.4 and the self-duality of $S(\mathbb{F}P)$, the top layer of the socle series of $S(\mathbb{F}P)$ is (isomorphic to) X and hence any nontrivial quotient of $S(\mathbb{F}P)$ has the top layer X. From (2.3) and Lemma 2.2, we have $\operatorname{Im}(Q|_{S(\mathbb{F}P)}) \cong S(\mathbb{F}P)/\operatorname{Ker}(Q|_{S(\mathbb{F}P)})$. It follows that X is the top layer of the socle series of $\operatorname{Im}(Q|_{S(\mathbb{F}P)}) \subseteq S(\mathbb{F}P^0)$. Inspecting the submodule lattice of $\mathbb{F}P^0$ given in Figure 2 of [ST], we see that $\operatorname{Im}(Q|_{S(\mathbb{F}P)})$ must be $(\mathbb{F} \oplus W) - Z - X$. In other words, $S(\mathbb{F}P)$ has a quotient isomorphic to $(\mathbb{F} \oplus W) - Z - X$.

By self-duality of $X = U_1'$ and $S(\mathbb{F}P)$, $U_1^{\perp}/U_1'$ is self-dual and has composition factors: Z (twice), $\mathbb{F}$, and W. If either $\mathbb{F}$ or W is a submodule of $U_1^{\perp}/U_1'$, it would be a direct summand in $U_1^{\perp}/U_1'$, which contradicts the conclusion of the previous paragraph. So Z must be the socle of $U_1^{\perp}/U_1'$ and therefore the socle series of $U_1^{\perp}/U_1'$ is $Z - (\mathbb{F} \oplus W) - Z$. Finally, we obtain the socle series of $S(\mathbb{F}P)$: $X - Z - (\mathbb{F} \oplus W) - Z - X$, as described. □

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