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**Advances in Geometry**


# A characterization of the *P*-geometry for $M_{23}$

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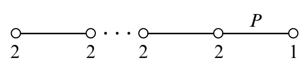
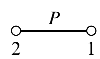
## 1 Introduction

The Petersen or *P*-geometries are diagram geometries belonging to the diagram



Here the final stroke  denotes the natural geometry of edges and vertices of the Petersen graph.

Five of the sporadic simple groups occur as flag-transitive automorphism groups of *P*-geometries of low rank. The books of Ivanov [3] and Ivanov and Shpectorov [5] are largely devoted to a proof that there are exactly eight flag-transitive *P*-geometries. It is thus desirable to find elementary geometric characterizations of these *P*-geometries. Additional conditions are necessary, since there is at least one *P*-geometry of rank 3 that is not flag-transitive, and the flag transitive *P*-geometries of rank 4 and 5 (the largest rank for which *P*-geometries are known to exist) admit huge numbers of distinct quotients that are not flag-transitive.

A first geometric classification was given in [2], where the two *P*-geometries of rank 3 that are flag-transitive, $\mathscr{G}(M_{22})$ and $\mathscr{G}(3 \cdot M_{22})$, were characterized among all rank 3 *P*-geometries as those that are linear and planar. The geometry is *linear* if any two lines meet in at most one point and *planar* if any three pairwise collinear points are coplanar. A third flag-transitive example is characterized in [1], where the *P*-geometry $\mathscr{G}(Co_2)$ is proven to be the unique linear and planar *P*-geometry of rank 4 whose *P*-residues of rank 3 are all $\mathscr{G}(M_{22})$.

With each *P*-geometry of rank $n$ we can associate a collection of geometries of rank $n-1$ called wide components. (See [3, p. 22, 311] and Proposition 2.1 below.) A crucial step in [2] was the identification of each wide component as either $\mathscr{G}(Sp_4(2))$ or $\mathscr{G}(3 \cdot Sp_4(2))$. The first case leads to $\mathscr{G}(M_{22})$ and the second to $\mathscr{G}(3 \cdot M_{22})$. This in turn implies that the wide components of an arbitrary *P*-geometry of rank $n$, each of

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whose rank 3 $P$-residues is $\mathscr{G}(M_{22})$, must be $C_{n-1}$ geometries of order 2. Therefore (see Theorem 2.3 below) each wide component is either a building $\mathscr{G}(Sp_{2n-2}(2))$ or the sporadic geometry $\mathscr{G}(A_7)$ for the alternating group of degree 7, in which case $n = 4$.

Here we show that the exceptional case characterizes a fourth flag-transitive $P$-geometry.

**Theorem 1.** *Suppose $\mathscr{G}$ is a $P$-geometry of rank* 4 *in which every point residue is isomorphic to $\mathscr{G}(M_{22})$ and every wide component is isomorphic to $\mathscr{G}(A_7)$. Then $\mathscr{G}$ is isomorphic to $\mathscr{G}(M_{23})$.*

The rest of this introduction provides references and notation. The second section presents and discusses the wide components of $P$-geometries. The third section contains the proof of the main theorem. A final section presents related problems of further interest.

Our general reference for diagram geometries is [6]. For tilde geometries, $P$-geometries, and their properties, see [3, 5]. The $P$-geometries $\mathscr{G}(M_{23})$ and $\mathscr{G}(M_{22})$ are described in [3, §3.4] in terms of the Witt design $W_{24}$ and its children $W_{23}$ and $W_{22}$ [3, §§2.3, 2.10]. The generalized quadrangle $\mathscr{G}(Sp_4(2))$ is described in [3, p. 59] and the sporadic geometry $\mathscr{G}(A_7)$ in [6, p. 152–153]. Also see [2] for $\mathscr{G}(Sp_4(2))$, $\mathscr{G}(3 \cdot Sp_4(2))$, $\mathscr{G}(M_{22})$, and $\mathscr{G}(3 \cdot M_{22})$.

By definition, geometries are residually connected. For us, an *incidence system* is a geometry without connectivity requirements.

Consider an incidence system $\mathscr{I}$ of rank $n \geqslant 2$ with a string diagram whose types are assigned consecutively from the interval $[1, n]$. For $I$ a subset of $[1, n]$, the set $\mathscr{I}_I$ consists of all elements of $\mathscr{I}$ with type belonging to $I$. The elements of $\mathscr{I}_1$, $\mathscr{I}_2$, $\mathscr{I}_3$, and $\mathscr{I}_4$ are, respectively, the *points*, *lines*, *planes*, and 3-*spaces* of $\mathscr{I}$. If $x$ has type $i$, then $\mathrm{res}^-_{\mathscr{I}}(x) = \mathrm{res}_{\mathscr{I}}(x) \cap \mathscr{I}_{[1,i-1]}$ and $\mathrm{res}^+_{\mathscr{I}}(x) = \mathrm{res}_{\mathscr{I}}(x) \cap \mathscr{I}_{[i+1,n]}$. When $\mathscr{I}$ is clear from the context, we may write res in place of $\mathrm{res}_{\mathscr{I}}$.

If $\mathscr{I}$ is a $P$-geometry, then a $P$-*residue* is $\mathrm{res}_{\mathscr{I}}(f) = \mathrm{res}^+_{\mathscr{I}}(f_i)$, for some flag $f$ of type $[1, i]$ with $i < n - 1$ and $f_i = f \cap \mathscr{I}_i$.

We thank Antonio Pasini for his help, particularly in guiding us to Theorem 2.3.

## 2 Wide components of $P$-geometries

In this section we discuss the existence and general properties of wide components for $P$-geometries. A reader interested only in the specific case considered in Theorem 1 could skip directly to the next section, taking the statements of Lemma 3.1 as given.

Let $\mathscr{G}$ be a $P$-geometry of rank $n$ at least 2. Let $\Sigma$ be the graph on the set $\mathscr{G}_{n-1}$ that has $a$ adjacent to $b$ provided there is a $p \in \mathscr{G}_{n-2}$ with the edges $a$ and $b$ opposite in the Petersen graph $\mathrm{res}^+(p)$. (So if $n = 2$, two elements of type $n - 1 = 1$ are adjacent in $\Sigma$ precisely when they are opposite edges of the Petersen graph.) The connected components in $\Sigma$ are then the *wide classes* in $\Sigma$. Especially, a wide class of the Petersen graph $\mathrm{res}^+(p)$ consists of three pairwise opposite edges. For $x \in \mathscr{G}_{[1,n-2]}$, each wide class of the $P$-geometry $\mathrm{res}^+_{\mathscr{G}}(x)$ is a connected subgraph of $\Sigma$.

For $i \neq n$ and $x \in \mathscr{G}_i$, let $X$ be a wide class of $\mathrm{res}^+(x)$ with $X = \{x\}$ if $i = n-1$. We define the *$i$-claw* $[x, X]$ to be the set of all $y \in \mathscr{G}_{[i,n-1]}$ for which there is some flag $\{x, y, z\}$ with $z \in X$. Thus $[x, X]$ can be thought of as cone-shaped, with *vertex* $x$ and *base* $X$, consisting of all $y$ of $\mathscr{G}_{[i,n-1]}$ on a geodesic from $x$ to $X$.

In particular $[x, X] \cap \mathscr{G}_{[1,i-1]} = \varnothing$, and $[x, X] \cap \mathscr{G}_i = \{x\}$. Furthermore $[x, X] \cap \mathscr{G}_{n-1} = X$, and an $n-1$-claw is just an element of $\mathscr{G}_{n-1}$.

The *claw system* $\mathscr{C} = \mathscr{C}(\mathscr{G})$ is the rank $n-1$ incidence system whose elements of type $i$ are the $i$-claws of $\mathscr{G}$ with incidence given by containment. That is, if $[x, X]$ is an $i$-claw and $[y, Y]$ is a $j$-claw for $i < j$, then they are incident precisely when $[x, X] \supset [y, Y]$.

We shall be interested in the type-preserving map $\mathrm{vert} : \mathscr{C} \to \mathscr{G}_{[1,n-1]}$ that takes each claw to its vertex: $\mathrm{vert}([x, X]) = x$.

The connected components of the claw system $\mathscr{C}(\mathscr{G})$ are the *wide components* of $\mathscr{G}$. (See [3, p. 311–312], especially [3, Lemma 7.1.7], for a similar construction.)

**Proposition 2.1.** *A wide component $\mathscr{W}$ is a string geometry of rank $n-1$ with $\mathscr{W}_{n-1} = \mathscr{W} \cap \mathscr{G}_{n-1}$ a union of wide classes of $\mathscr{G}$. For $x \in \mathscr{W}_{n-1}$, the restriction* $\mathrm{vert} : \mathrm{res}_{\mathscr{W}}(x) \to \mathrm{res}^-_{\mathscr{G}}(x)$ *is an isomorphism of rank $n-2$ projective spaces over* $\mathbb{F}_2$.

*Proof.* The geometry $\mathscr{G}$ contains flags of rank $n$, and incidence is given by containment; so $\mathscr{C}$ is a string incidence system of rank $n-1$. By definition, $a$ and $b$ are adjacent in $\Sigma$ when there is in $\mathscr{C}$ an $n-2$-claw $[p, \{a, b, c\}]$. Therefore $a$ and $b$ are adjacent in $\Sigma$ if and only if they have distance 2 in the bipartite graph $\mathscr{C}_{n-2} \cup \mathscr{C}_{n-1}$, and any wide class of $\mathscr{G}$ is in a single wide component.

Let $x \in \mathscr{W}_{n-1}$. For each $w \in \mathrm{res}^-_{\mathscr{G}}(x)$, there is a unique claw $[w, W]$ in $\mathrm{res}_{\mathscr{W}}(x)$, namely that for which $W$ is the wide class of $\mathrm{res}_{\mathscr{G}}(w)$ containing $x$. Therefore $\mathrm{vert} : \mathrm{res}_{\mathscr{W}}(x) \to \mathrm{res}^-_{\mathscr{G}}(x)$ is a bijection. If $[w, W]$ and $[y, Y]$ are claws of $\mathrm{res}_{\mathscr{W}}(x)$, then $x \in W \cap Y$ is nonempty. If additionally $w$ and $y$ are incident, then this implies that $W \subseteq Y$ or $W \supseteq Y$. Therefore $[w, W]$ and $[y, Y]$ are incident in $\mathrm{res}_{\mathscr{W}}(x)$ if and only if $w$ and $y$ are incident in $\mathrm{res}^-_{\mathscr{G}}(x)$. In the $P$-geometry $\mathscr{G}$, we see that $\mathrm{res}^-_{\mathscr{G}}(x)$ is an $A_{n-2}$ geometry of over $\mathbb{F}_2$; so we have verified the second sentence.

The proof will be complete once we have shown that $\mathscr{W}$ is residually connected. In view of the previous paragraph, it suffices to show that $\mathrm{res}_{\mathscr{W}}([x, X])$ is connected, for $x$ a point of $\mathscr{G}$ and $X$ a wide class of $\mathrm{res}_{\mathscr{G}}(x)$. Let $[a, A]$ and $[b, B]$ be two claws of $\mathrm{res}_{\mathscr{W}}([x, X])$. Thus $a$ and $b$ are incident to $x$, and $A$ and $B$ are contained in $X$. For $a_0 \in A$ and $b_0 \in B$, we have $[a, A]$ connected to $a_0$ and $[b, B]$ connected to $b_0$ in $\mathrm{res}_{\mathscr{W}}([x, X])$. We apply the final sentence of the first paragraph within the $P$-geometry $\mathrm{res}_{\mathscr{G}}(x)$ to find that $a_0$ and $b_0$ are connected in $\mathrm{res}_{\mathscr{W}}([x, X])$. Therefore $[a, A]$ and $[b, B]$ are connected in $\mathrm{res}_{\mathscr{W}}([x, X])$, as desired. □

**Proposition 2.2.** *Let $\mathscr{G}$ be a $P$-geometry of rank $n \geqslant 4$.*

(1) *If* $\mathrm{res}^+_{\mathscr{G}}(x)$ *is* $\mathscr{G}(M_{22})$, *for all* $x \in \mathscr{G}_{n-3}$, *then each wide component* $\mathscr{W}$ *is a* $C_{n-1}$ *geometry of order* 2. (*That is, each rank* 2 *residue of the* $C_{n-1}$ *geometry* $\mathscr{W}$ *is defined over* $\mathbb{F}_2$.)

(2) *If* $\mathrm{res}^+_{\mathscr{G}}(x)$ *is* $\mathscr{G}(3 \cdot M_{22})$, *for all* $x \in \mathscr{G}_{n-3}$, *then each wide component* $\mathscr{W}$ *is a rank* $n-1$ *tilde geometry*.

(V7 4/5/05 17:03) WDG/G J-1297 Adv. in Geom., 5:3 PMU: L(R) 18/04/05 Tmath .3.05.05 pp. 429-438 006_P (p. 431)

(3) *If we are in Case* (1) *or in Case* (2), *then the restriction* vert : $\mathscr{W} \to \mathrm{vert}(\mathscr{W}) \subseteq \mathscr{G}_{[1,n-1]}$ *is a* 2*-cover of incidence systems.*

*Proof.* By definition, the restriction of vert to $\mathscr{W}$ is a 2-cover of vert($\mathscr{W}$) when vert induces an isomorphism of each rank 2 residue of $\mathscr{W}$ with its image in $\mathscr{G}_{[1,n-1]}$. If $f$ is a flag in $\mathscr{W}$ of cotype $\{i, j\}$, then by Proposition 2.1 vert is an isomorphism on $\mathrm{res}_{\mathscr{W}}(f)$, except possibly when $\{i, j\} = \{n-2, n-1\}$. In this remaining case, let $[x, X] \in f \cap \mathscr{W}_{n-3}$. Then within $\mathrm{res}^+_{\mathscr{G}}(x)$ we can check that vert gives an isomorphism of $\mathrm{res}^+_{\mathscr{W}}([x, X])$ with a wide component of $\mathrm{res}^+_{\mathscr{G}}(x)$. The wide component is of type $\mathscr{G}(Sp_4(2))$ under Case (1) and $\mathscr{G}(3 \cdot Sp_4(2))$ under Case (2). □

The 2-cover vert is an isomorphism of $\mathscr{W}$ with an induced subsystem (an *embedding* of $\mathscr{W}$) precisely when it is injective. This is equivalent to requiring that the image system be a subgeometry, since injectivity only fails if, for some element $x$, there is more than one wide class in $\mathrm{res}^+_{\mathscr{W}}(x)$.

In all the flag-transitive examples, vert is an embedding of $\mathscr{W}$ as a subgeometry of $\mathscr{G}$.

Recall that the sporadic geometry $\mathscr{G}(A_7)$ is a $C_3$ geometry with seven points, 35 lines (being all 3-sets of points), and 15 planes (one orbit of $A_7$ acting on projective planes based on the point set).

**Theorem 2.3.** *For $m \geqslant 2$, a $C_m$ geometry of order* 2 *is either a building $\mathscr{G}(Sp_{2m}(2))$ or the sporadic geometry $\mathscr{G}(A_7)$ for the alternating group of degree* 7, *in which case $m = 3$.*

*Proof.* This can be extracted almost entirely from the book of Pasini [6].

As the geometry has order 2, it is locally finite and so finite by [6, Corollary 7.32]. If $m = 2$, we have $\mathscr{G}(Sp_4(2))$. If $m \geqslant 3$, then the $C_3$-residue of the geometry is a polar space or flat by [6, Theorem 14.17]. Therefore if $m \geqslant 4$, the geometry is a polar space by [6, Corollary 14.19]. We conclude that a $C_m$ geometry of order 2 either is a polar space or is flat and $m = 3$. In the first case it must be $\mathscr{G}(Sp_{2m}(2))$, for instance, by Shult's Triangle Theorem [8]. In the second case, it must be $\mathscr{G}(A_7)$ by a result of Rees [7, Lemma 5.14]. □

**Proposition 2.4.** *Let $\mathscr{G}$ be a P-geometry of rank $n \geqslant 4$, and let $\mathscr{W}$ be a wide component of $\mathscr{G}$. If $\mathrm{res}^+_{\mathscr{G}}(x)$ is $\mathscr{G}(M_{22})$, for all $x \in \mathscr{G}_{n-3}$, then the set $\mathscr{W}_{n-1}$ is a wide class of $\mathscr{G}$.*

*Proof.* As seen above, adjacency in $\Sigma$ corresponds to distance 2 in $\mathscr{C}_{n-2} \cup \mathscr{C}_{n-1}$. The $C_{n-1}$ geometry $\mathscr{W}$ is one of those given in Theorem 2.3, so within it we check that $\mathscr{W}_{n-2} \cup \mathscr{W}_{n-1}$ is connected. As $\mathscr{W}_{n-1}$ is a union of wide classes of $\mathscr{G}$ by Proposition 2.1, it is in fact a single wide class. □

The proposition suggests that in Case (1) of Proposition 2.2 the map vert is always an embedding. In the main case of interest to us, this holds.

**Lemma 2.5.** *Let $\mathscr{G}$ be a $P$-geometry of rank* 4. *If* $\mathrm{res}^+_{\mathscr{G}}(x)$ *is* $\mathscr{G}(M_{22})$, *for all* $x \in \mathscr{G}_{n-3}$, *and the wide component* $\mathscr{W}$ *is of type* $\mathscr{G}(A_7)$, *then the map* $\mathrm{vert} : \mathscr{W} \to \mathscr{G}$ *is an embedding of* $\mathscr{W}$ *in* $\mathscr{G}$.

*Proof.* We must show that vert is injective on $\mathscr{W}_i$ for $i = 1, 2, 3$. This is immediate for $\mathscr{W}_3$, a subset of $\mathscr{G}_3$ of size 15.

Next consider a point $p$ with $[p, P] \in \mathscr{W}_1$, and let $x \in P$. Examine $\mathrm{res}_{\mathscr{G}}(p)$, isomorphic to $\mathscr{G}(M_{22})$. The wide class $A$ of this residue that contains $x$ must be contained in $P$. But wide components of $\mathscr{G}(M_{22})$ have type $\mathscr{G}(Sp_4(2))$, so $|A| = 15$. As $A \subseteq P \subseteq \mathscr{W}_3$, we find $A = P = \mathscr{W}_3$. In particular, for all $[p_1, P_1]$ and $[p_2, P_2]$ in $\mathscr{W}_1$, we have $P_1 = \mathscr{W}_3 = P_2$. Thus $[p_1, P_1] = [p_2, P_2]$ if and only if $p_1 = p_2$, and vert is injective on $\mathscr{W}_1$.

Finally, suppose that $[L, B_1], [L, B_2] \in \mathscr{W}_2$. We may assume that $p \in L$. But then $B_1 \cup B_2 \subseteq \mathscr{W}_3 = P$, so $[L, B_1]$ and $[L, B_2]$ are incident to $[p, P]$ and in a single wide component of $\mathrm{res}_{\mathscr{G}}(p)$. Such a wide component has type $\mathscr{G}(Sp_4(2))$, within which we see that $B_1 = B_2$. Therefore vert is injective on $\mathscr{W}_2$ as well, giving the lemma. □

## 3 Proof of Theorem 1

Let $\mathscr{G}$ be a $P$-geometry of rank 4 as in the hypothesis of Theorem 1. We identify each wide component (or component, for short) with the corresponding subgeometry of $\mathscr{G}$ (as we may, by Lemma 2.5). In our situation, the results of the previous section give:

**Lemma 3.1.** (1) *Every wide component* $\mathscr{W}$ *is an induced subgeometry of points, lines, and planes of* $\mathscr{G}$ *that is isomorphic to* $\mathscr{G}(A_7)$.

(2) *Every plane of* $\mathscr{G}$ *belongs to a unique wide component of* $\mathscr{G}$.

(3) *If the line $L$ of $\mathscr{G}$ belongs to the wide component $\mathscr{W}$, then the three planes of* $\mathrm{res}^+_{\mathscr{W}}(L)$ *form a triple of pairwise opposite edges in the Petersen graph* $\mathrm{res}^+_{\mathscr{G}}(L)$.

A reader who has chosen not to read the previous section can take Lemma 3.1 as defining the set of wide components in $\mathscr{G}$.

Suppose $u$ and $v$ are collinear points of $\mathscr{G}$. We first study $\mathrm{res}(u) \cap \mathrm{res}(v)$.

**Lemma 3.2.** *No two lines in* $\mathrm{res}(u) \cap \mathrm{res}(v)$ *are coplanar*.

*Proof.* In a plane two lines share only one point. □

Let $L_1, L_2, \dots, L_m$ be the lines from $\mathrm{res}(u) \cap \mathrm{res}(v)$, and let the wide components containing both $u$ and $v$ be $\mathscr{W}_1, \mathscr{W}_2, \dots, \mathscr{W}_n$.

**Lemma 3.3.** *Every $L_i$ is contained in exactly five components $\mathscr{W}_j$ and every $\mathscr{W}_j$ contains exactly five lines $L_i$ on $u$ and $v$. In particular, $m = n$.*

*Proof.* The first claim follows from Lemma 3.1.2–3. Since the component $\mathscr{W}_j$ is isomorphic to $\mathscr{G}(A_7)$, we get that $u$ and $v$ are incident to exactly five lines in $\mathscr{W}_j$. □

We examine $\mathrm{res}(u) \cap \mathrm{res}(v)$ in terms of its embedding in $\mathrm{res}(u)$, which is isomorphic to $\mathscr{G}(M_{22})$. Recall that under this isomorphism the lines from $\mathrm{res}(u)$ correspond to pairs of points of $W_{22}$, the unique 3-$(22, 6, 1)$ design [9, Satz 4]. (To avoid confusion between points of $\mathscr{G}$ and points of $W_{22}$, we call the latter *Witt points*.) Two lines of $\mathrm{res}(u)$ are coplanar precisely when the corresponding pairs are disjoint and together in a hexad of $W_{22}$. The wide components containing $u$ correspond to hexads. A line of $\mathscr{G}$ lies in a component if and only if the corresponding pair lies in the corresponding hexad.

**Lemma 3.4.** *We have $n \geqslant 21$.*

*Proof.* Since two hexads of $W_{22}$ cannot meet in more than two Witt points, two components containing $u$ share at most one line on $u$. As $u$ and $v$ are collinear, there is at least one line $L_i$. By the previous lemma, this line is contained in five components. Each of these contains four additional lines on $u$ and $v$. Any two of the hexads among those corresponding to the five components meet only in the pair corresponding to $L_i$. Therefore, no two of the new lines coincide, and hence we have at least 21 lines on $u$ and $v$. □

According to Lemma 3.2, two lines $L_i$ and $L_j$ that lie in the same component $\mathscr{W}_k$ are never coplanar. As the corresponding pairs in $\mathrm{res}(u)$ are in a common hexad, they must share a Witt point. Since this is true for any two of the five lines $L_i$ in $\mathscr{W}_k$, the five associated pairs in the hexad $H_k$ of $\mathrm{res}(u)$ that corresponds to $\mathscr{W}_k$ must have a common Witt point, which we call the *base point* of $H_k$.

**Lemma 3.5.** *For all $i$ and $j$, the base point of $H_i$ lies in $H_j$.*

*Proof.* Suppose the base point $a$ of $H_i$ is not contained in $H_j$. Let $b$ be the base point of $H_j$.

We claim that $b \notin H_i$. Indeed, if $b \in H_i$ then $H_i \cap H_j = \{b, c\}$ for some $c$. Since $b$ is the base point of $H_j$, the pair $\{b, c\}$ corresponds to some line $L_k$. As $\{b, c\}$ is also contained in $H_i$, it must contain the base point of $H_i$, that is, $a$, a contradiction. Thus, $b \notin H_i$, as claimed.

Let $c$, $(c \neq a)$ be any Witt point in $H_i \backslash H_j$; and let $H$ be the unique hexad containing $a$, $b$, and $c$. Then $H \cap H_i = \{a, c\}$ and $H \cap H_j = \{b, d\}$ for some $d$. Since $c \notin H_j$, we have $c \neq d$. Thus, the pairs $\{a, c\}$ and $\{b, d\}$ of $H$ are disjoint, hence the corresponding lines $L_k$ are coplanar, against Lemma 3.2. □

**Corollary 3.6.** *We have $n = 21$. The 21 hexads $H_i$ have the same base point $a$, and the 21 lines $L_i$ correspond to the pairs containing $a$. These 21 pairs $\{a, x\}$ carry the structure of a projective plane of order 4, where a projective line consists of five pairs within a given hexad $H_i$.*

*Proof.* By the lemma every base point is contained in the intersection of all the $H_i$. Since there are at least 21 $H_i$'s, there is only one base point, which we call $a$. As there are exactly 21 hexads on a fixed Witt point, the $H_i$ exhaust the hexads on $a$. The final sentence follows from an examination of $W_{22}$ and also can be verified directly. □

Corollary 3.6 establishes a mapping from the set of points $v$ of $\mathcal{G}$ that are collinear with $u$ to the set of Witt points of $\mathrm{res}(u)$, given by $v \mapsto a = a_v$, the uniquely determined base point of $\mathrm{res}(u) \cap \mathrm{res}(v)$.

**Lemma 3.7.** *The mapping $v \mapsto a_v$ is a bijection. In particular, every point $u$ of $\mathcal{G}$ is collinear with exactly* 22 *other points.*

*Let the line $L$ be incident to the points $\{u, v, w\}$. Then in* $\mathrm{res}(u)$ *the line $L$ corresponds to the pair $\{a, b\}$ of Witt points if and only if $\{a, b\} = \{a_v, a_w\}$.*

*Proof.* We first show that this mapping is an injection. Suppose $v$ and $w$ are two points of $\mathcal{G}$ that are collinear with $u$ and correspond to the same base point $a$. Let $\{a, b\}$ and $\{a, c\}$ be distinct pairs, and let $L$ and $L'$ be the lines on $u$ corresponding to these pairs. Then $L$ and $L'$ are incident to the same three points, $u$, $v$ and $w$. Let $H$ be the hexad containing $a$, $b$, and $c$. Then $L$ and $L'$ are lines of the wide component $\mathcal{W}$ of $H$. Since no two lines of $\mathcal{W}$, isomorphic to $\mathcal{G}(A_7)$, have the same three points, we get a contradiction which proves injectivity. Surjectivity will follow from the final sentence of the lemma.

Let $L$ be the line on $u$ corresponding to $\{a, b\}$, and let $v$ and $w$ be the two remaining points incident to $L$. Since $\mathrm{res}(u) \cap \mathrm{res}(v)$ and $\mathrm{res}(u) \cap \mathrm{res}(w)$ cannot have the same base point, we must have $\{a, b\} = \{a_v, a_w\}$, as desired. □

**Lemma 3.8.** *The collinearity graph of $\mathcal{G}$ is complete of size* 23. *Furthermore, for every triple of points of $\mathcal{G}$ there is a unique line incident to this triple.* (*In particular, lines can be identified with their point sets.*)

*Proof.* Suppose $v$ and $w$ are points of $\mathcal{G}$ collinear with $u$. Let the base point of $v$ in $\mathrm{res}(u)$ be $a$ and the base point of $w$ be $b$. Then the line $L$ on $u$ corresponding to $\{a, b\}$ has points $u$, $v$, and $w$, as in Lemma 3.7. This shows that $v$ and $w$ are collinear, proving that the collinearity graph is complete. It also shows that for every triple of points $\{u, v, w\}$ there is a line through them. This line is unique since pairs corresponding to distinct lines are distinct. □

Consider the block design $\mathcal{D}$ on the point set of $\mathcal{G}$, where a block consists of seven points lying in a wide component.

**Lemma 3.9.** *The design $\mathcal{D}$ is isomorphic to the Witt design $W_{23}$.*

*Proof.* By the theorem of Witt [9, Satz 5] (or see [3, Lemma 2.9.4]), it suffices to show that any four points $u$, $v$, $w$, and $x$ lie in a unique wide component. This is clear, since in $\mathrm{res}(u)$ there is a unique hexad containing the base points of $v$, $w$, and $x$. □

To complete the identification of $\mathcal{G}$ it remains to describe its elements and incidence in terms of $\mathcal{D}$. In fact, it will be more convenient to work in the Witt design

$W_{24}$, which we obtain by adding an extra point $\infty$. Then $\mathscr{D}$ is the residue design of $\infty$ in $W_{24}$, and the blocks of $\mathscr{D}$ are obtained by removing $\infty$ from the octads of $W_{24}$ containing it. We remark that the remaining octads are symmetric differences of two blocks of $\mathscr{D}$ meeting in three points, and so they are determined by $\mathrm{res}(u)$ as well.

By Lemma 3.8, the points of $\mathscr{G}$ are the points of $\mathscr{D}$, those points of $W_{24}$ not equal to $\infty$. Wide components are flat; that is, every plane in a wide component is incident with all seven points of the component. Thus, all planes of a wide component have the same points. Indeed, by Lemma 3.9 two planes have the same point set if and only if they belong to the same component.

Again by Lemma 3.8, the lines of $\mathscr{G}$ are the triples of points of $\mathscr{D}$ with incidence being containment. Equally well, we can think of a line as a tetrad (4-set) containing $\infty$. Each such tetrad in turn lies in a unique sextet, a partition of the 24 points of $W_{24}$ into six tetrads, any two of which have union an octad. A trio is a partition into three octads. We can also view octads as partitions with two parts, the octad and its complement. Two partitions are said to be *compatible* if one of them is a refinement of the other.

**Lemma 3.10.** (1) *The* 3-*spaces of* $\mathscr{G}$ *correspond bijectively to octads of* $W_{23}$. *A point is incident to a* 3-*space if and only if it does not belong to the corresponding octad.*

(2) *A line is incident to a* 3-*space if and only if the corresponding sextet and octad are compatible.*

*Proof.* The number of 3-spaces in $\mathscr{G}$ is equal to the number of octads in $W_{23}$, namely 506. (The first can be calculated by counting incident pairs of points and 3-spaces in two ways, and the second by subtracting the number of blocks in $W_{23}$ from the number of octads in $W_{24}$.) Therefore to prove (1) it suffices to show that the point set of a 3-space is the complement of an octad and that the complement of every octad is the point set of a 3-space. This we do in the course of proving (2).

We next show that if the line $L$ is incident to the 3-space $V$, then the point set of $V$ is the complement of an octad that is compatible with the sextet of $L$. This gives one direction of (2) and shows that the point set of any 3-space is the complement of an octad.

The line $L$ is in five wide components by Lemma 3.3. The point set of each of these components coincides with some $L \cup T_i$, where the sextet of $L$ consists of $\{\infty\} \cup L$ and $T_i$, for $1 \leqslant i \leqslant 5$. The 3-space $V$ is incident to three planes of $\mathrm{res}^+(L)$. By Lemma 3.1.3, those three planes belong to three different wide components, since the corresponding edges of the Petersen graph share a Petersen vertex, namely $V$. This shows that the point set of $V$, being the union of the point sets of its three planes on $L$, is the union of $L$ and three of the tetrads $T_i$. Therefore, the complement in $\mathscr{D}$ of the point set of $V$ is one of the octads $T_i \cup T_j$, as desired.

Since every octad of $W_{23}$ can be realized as $T_i \cup T_j$, for some choice of line $L$ and corresponding sextet, it remains to prove that to each $T_i \cup T_j$ there is indeed a corresponding 3-space incident to $L$. The line $L$ is incident to ten different 3-spaces (the number of vertices in the Petersen graph $\mathrm{res}^+(L)$), and there are exactly ten octads $T_i \cup T_j$. If two 3-spaces, $V_1$ and $V_2$, are associated with the same octad, say $T_4 \cup T_5$, then the three planes on $L$ and each of the $V_i$, although possibly different, must have point sets $L \cup T_1$, $L \cup T_2$, and $L \cup T_3$. Recall that two planes have the same point set precisely when they belong to the same wide component. Therefore, in the Petersen graph $\mathrm{res}^+(L)$, the vertices $V_1$ and $V_2$ are incident to members of the same three sets of opposite edges. This forces $V_1 = V_2$, as required. □

**Lemma 3.11.** *The planes of $\mathcal{G}$ correspond bijectively to trios. The points of a plane are those of the same octad in the trio as $\infty$. The lines of the plane correspond to the sextets compatible with the trio. A plane and* 3*-space are incident when the corresponding trio contains the corresponding octad.*

*Proof.* Let $P$ be a plane, $L$ a line incident to the plane, and $V_1$ and $V_2$ the two 3-spaces incident to $P$. By the previous lemma, these two spaces correspond to distinct octads, $O_1$ and $O_2$, both compatible with the sextet $S$ of $L$. If $O_1$ and $O_2$ were not disjoint, then $O_1 \cap O_2$ would be a tetrad belonging to the sextet of every line in the plane, which is not the case. Therefore each plane corresponds to a pair of disjoint octads, each compatible with the sextet of each line of the plane. The octads $O_1$ and $O_2$ together with their complementary octad $O_3$ form a trio compatible with $S$, the points of $P$ being $O_3 \backslash \infty$. There are exactly seven sextets compatible with a given trio, and here these must correspond to the seven lines of $P$. In particular, this trio corresponds to a unique plane.

There are three octads of $W_{23}$ compatible with $S$ and disjoint from the octad $O_1$. These correspond to the three planes of $V_1$ containing $L$. Therefore every trio compatible with $S$ corresponds to a plane incident to $L$, hence every trio of $W_{23}$ corresponds to a unique plane. □

This lemma completes the proof of Theorem 1. In summary, we have identified the points, lines, planes, and 3-spaces of $\mathcal{G}$ as, respectively, the points of $W_{24}$ other than $\infty$, the sextets, the trios, and the octads not on $\infty$. A point is incident to the sextets and trios in which it shares a tetrad or octad with $\infty$ and to the octads not containing it (and $\infty$). Sextets, trios, and octads are incident when compatible.

Thus our description of $\mathcal{G}$ matches the description of $\mathcal{G}(M_{23})$ given in [3, p. 115–116].

## 4 Problems

**Problem 4.1.** *Prove*: *a rank* 3 *$P$-geometry with all wide components equal to $\mathcal{G}(Sp_4(2))$ must be $\mathcal{G}(M_{22})$.* (*In that case, the $P$-residue hypothesis of Theorem* 1 *would not be necessary.*)

**Problem 4.2.** *In a linear rank* 4 *$P$-geometry with all rank* 3 *$P$-residues $\mathcal{G}(M_{22})$, all wide components are $\mathcal{G}(Sp_6(2))$ by Theorem* 2.2. *Prove*: *planarity follows from this* (*so that* [1] *applies*).

**Problem 4.3.** *Prove*: *in a rank* 4 *$P$-geometry with all rank* 3 *$P$-residues $\mathcal{G}(M_{22})$, either all wide components are $\mathcal{G}(A_7)$ or all wide components are $\mathcal{G}(Sp_6(2))$.*

**Problem 4.4.** *Prove*: *a rank* 5 *linear and planar* $P$-*geometry with all rank* 4 $P$-*residues* $\mathcal{G}(Co_2)$ (*or*, *perhaps*, *all rank* 3 $P$-*residues* $\mathcal{G}(M_{22})$) *is* $\mathcal{G}(BM)$. (*See* [4] *and* [5, p. 273–274].)

**Problem 4.5.** *Characterize each of the flag-transitive Petersen geometries by its wide component and rank* 3 $P$-*residue* (*but see Problem* 4.1).

**Problem 4.6.** *Characterize each of the flag-transitive tilde geometries by its wide component and rank* 3 $P$-*residue*.

**Problem 4.7.** *Under what circumstances is the map* vert *of the second section an embedding*?

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