Introduction to Lie Algebras

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Part I

Introduction

Chapter 1

Introduction

1.1 Algebras

Let $\mathbb{K}$ be a field. A $\mathbb{K}$-*algebra* $({}_{\mathbb{K}}A, \mu)$ is a (left) $\mathbb{K}$-space A equipped with a bilinear multiplication. That is, there is a $\mathbb{K}$-space homomorphism *multiplication* $\mu\colon A \otimes_{\mathbb{K}} A \longrightarrow A$. We often write ab in place of $\mu(a \otimes b)$. Also we may write A or (A, μ) in place of $({}_{\mathbb{K}}A, \mu)$ when the remaining pieces should be evident from the context.

If A is a $\mathbb{K}$-algebra, then its *opposite algebra* A^{op} has the same underlying vector space but its multiplication μ^{op} is given by $\mu^{\mathrm{op}}(x \otimes y) = \mu(y \otimes x)$.

(1.1). LEMMA. *The map $\mu\colon A \otimes_{\mathbb{K}} A \mapsto A$ is a $\mathbb{K}$-algebra multiplication if and only if the* adjoint map

$$\mathrm{ad}\colon x \mapsto \mathrm{ad}_x \quad \textit{given by} \quad \mathrm{ad}_x\, a = xa$$

is a $\mathbb{K}$-vector space homomorphism of A into $\mathrm{End}_{\mathbb{K}}(A)$ *.* □

If $\mathcal{V} = \{\, v_i \mid i \in I \,\}$ is a $\mathbb{K}$-basis of A, then the algebra is completely described by the associated *multiplication coefficients* or *structure constants* $c_{ij}^k \in \mathbb{K}$ given by

$$v_i v_j = \sum_{k \in I} c_{ij}^k v_k\,,$$

for all i, j.

We may naturally *extend scalars* from $\mathbb{K}$ to any extension field $\mathbb{E}$. Indeed $\mathbb{E} \otimes_{\mathbb{K}} A$ has a natural $\mathbb{E}$-algebra structure with the same multiplication coefficients for the basis $\mathcal{V}$.

Going the other direction is a little more subtle. If the $\mathbb{E}$-algebra B has a basis $\mathcal{V}$ for which all the c_{ij}^k belong to $\mathbb{K}$, then the $\mathbb{K}$-span of the basis is a $\mathbb{K}$-algebra A for which $B = \mathbb{E} \otimes_{\mathbb{K}} A$. In that case we say that A is a $\mathbb{K}$-*form* of the algebra B. In many cases the $\mathbb{E}$-algebra B has several pairwise nonisomorphic $\mathbb{K}$-forms.

Various generalizations of the above are available and often helpful. The extension field $\mathbb{E}$ of $\mathbb{K}$ is itself a special sort of $\mathbb{K}$-algebra. If C is an arbitrary $\mathbb{K}$-algebra, then $C \otimes_{\mathbb{K}} A$ is a $\mathbb{K}$-algebra, with opposite algebra $A \otimes_{\mathbb{K}} C$. The relevant multiplication is $\mu = \mu_C \otimes \mu_A$:

$$\mu((c_1 \otimes a_1) \otimes (c_2 \otimes a_2)) = \mu_C(c_1 \otimes c_2) \otimes \mu_A(a_1 \otimes a_2)\,.$$

We might also wish to consider R-algebras for other rings R with identity. For the tensor product to work reasonably, R should be commutative. A middle ground would require R to be an integral domain, although even in that case we must decide whether or not we wish algebras to be free as R-module.

Of primary interest to us is the case $R = \mathbb{Z}$. A $\mathbb{Z}$-*algebra* is a free abelian group (that is, *lattice*) $L = \bigoplus_{i \in I} \mathbb{Z}v_i$ provided with a bilinear multiplication $\mu_{\mathbb{Z}}$ which is therefore completely determined by the integral multiplication coefficients c_{ij}^k. From this we can construct $\mathbb{K}$-algebras $L_{\mathbb{K}} = \mathbb{K} \otimes_{\mathbb{Z}} L$ for any field $\mathbb{K}$, indeed for any $\mathbb{K}$-algebra. For instance $C \otimes_{\mathbb{Z}} \mathrm{Mat}_n(\mathbb{Z})$ is the $\mathbb{K}$-algebra $\mathrm{Mat}_n(C)$ of all $n \times n$ matrices with entries from the $\mathbb{K}$-algebra C.

Suppose for the basis $\mathcal{V}$ of the $\mathbb{K}$-algebra A all the c_{ij}^k are integers—that is, belong to the subring of $\mathbb{K}$ generated by 1. Then the $\mathbb{Z}$-algebra $L = \bigoplus_{i \in I} \mathbb{Z}v_i$ with these multiplication coefficients can be viewed as a $\mathbb{Z}$-*form* of A (although we only have its quotient by $\mathrm{char}(\mathbb{K})$ as a subring of A). The original $\mathbb{K}$-algebra A is then isomorphic to $L_{\mathbb{K}}$.

1.2 Types of algebras

As $\dim_{\mathbb{K}}(A \otimes_{\mathbb{K}} A) \geq \dim_{\mathbb{K}}(A)$, every $\mathbb{K}$-space admits $\mathbb{K}$-algebras. We focus on those with some sort of interesting additional structure. Examples are associative algebras, Jordan algebras, alternative algebras, composition algebras, Hopf algebras, and Lie algebras—these last being the primary focus of our study. (All the others will be discussed at least briefly.)

In most cases these algebra types naturally form subcategories of the additive category ${}_{\mathbb{K}}\mathsf{Alg}$ of $\mathbb{K}$-algebras, the maps φ of $\mathrm{Hom}_{{}_{\mathbb{K}}\mathsf{Alg}}(A, B)$ being those linear transformations $\varphi \in \mathrm{Hom}_{\mathbb{K}}(A, B)$ with $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in A$. As the category ${}_{\mathbb{K}}\mathsf{Alg}$ is additive, each morphism has a kernel and image, which are defined as usual and enjoy the usual properties.

A subcategory will often be defined initially as belonging to a particular *variety* of $\mathbb{K}$-algebras. For instance, the *associative* $\mathbb{K}$-*algebras* are precisely those $\mathbb{K}$-algebras satisfying the identical relation

$$(xy)z = x(yz)\,.$$

Alternatively, the associative $\mathbb{K}$-algebras are those whose multiplication map μ satisfies

$$\mu(\mu(x \otimes y) \otimes z) = \mu(x \otimes \mu(y \otimes z))\,.$$

As the defining identical relation is equivalent to its reverse $(zy)x = z(yx)$, the opposite of an associative algebra is also associative.

Similarly, the subcategory of *alternative* $\mathbb{K}$-*algebras* is the variety of $\mathbb{K}$-algebras given by the weak associative laws

$$x(xy) = (xx)y \quad \text{and} \quad y(xx) = (yx)x\,.$$

The opposite of an alternative algebra is also alternative.

Varietal algebras like these have nice local properties:

(i) A $\mathbb{K}$-algebra is associative if and only if all its 3-generator subalgebras are associative.

(ii) A $\mathbb{K}$-algebra is alternative if and only if all its 2-generator subalgebras are alternative.

The associative identity is linear in that each variable appears at most once in each term, while the alternative identity is not, since x appears twice in each term. The linearity of an identity implies that it only need be checked on a basis of the algebra to ensure that it is valid throughout the algebra. That is, there are appropriate identities among the various c_{ij}^k that are equivalent to the algebra being associative. (Exercise: find them.) This implies the (admittedly unsurprising) fact that extending the scalars of an associative algebra produces an associative algebra. It is also true that extending the scalars of an alternative algebra produces another alternative algebra, but that needs some discussion since the basic identity is not linear. (Exercise.)

The basic model for an associative algebra is $\mathrm{End}_{\mathbb{K}}(V)$ for some $\mathbb{K}$-space V. Indeed, most associative algebras (including all with an identity) are isomorphic to subalgebras of various $\mathrm{End}_{\mathbb{K}}(V)$. (See Proposition (1.3).) For finite dimensional V we often think in matrix terms by choosing a basis for V and then using that basis to define an isomorphism of $\mathrm{End}_{\mathbb{K}}(V)$ with $\mathrm{Mat}_n(\mathbb{K})$ for $n = \dim_{\mathbb{K}}(V)$.

Of course, every associative algebra is alternative, but we now construct the most famous models for alternative but nonassociative algebras. If we start with $\mathbb{K} = \mathbb{R}$, then we have the familiar construction of the complex numbers as 2×2 matrices: for $a, b \in \mathbb{K}$ we set

$$(a, b) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

with multiplication given by

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac - bd & ad + bc \\ -bc - ad & -bd + ac \end{pmatrix}$$

and conjugation given by

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}^{-} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

As $\mathbb{R}$ is commutative and conjugation is trivial on $\mathbb{R}$, these can be rewritten:

For $a, b \in \mathbb{K}$ and $a \mapsto \bar{a}$ an antiautomorphism of $\mathbb{K}$, we set

$$(a,b)_{\mathbb{K}} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

with

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} c & d \\ -\bar{d} & \bar{c} \end{pmatrix} = \begin{pmatrix} ac - \bar{d}b & da + b\bar{c} \\ -c\bar{b} - \bar{a}\bar{d} & -\bar{b}d + \bar{c}\bar{a} \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}^{-} = \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix}.$$

This then gives us the complex numbers $\mathbb{C}$ as the collection of all pairs $(a,b)_{\mathbb{R}}$ of real numbers. Feeding the complex numbers back into the machine produces Hamilton's *quaternions* $\mathbb{H}$ as all pairs $(a,b)_{\mathbb{C}}$ with the multiplication and the conjugation antiautomorphism described. As $\mathbb{C}$ is commutative the quaternions are associative, but they are no longer commutative.

Finally with $\mathbb{K} = \mathbb{H}$, the resulting $\mathbb{O}$ of all pairs $(a,b)_{\mathbb{H}}$ is the *octonions* of Cayley and Graves. The octonions are indeed alternative but not associative, although this requires checking. Again conjugation is an antiautomorphism.

In each case, the 2×2 "scalar matrices" are only those with $b = 0$ and $a = \bar{a} \in \mathbb{R}$, so we have constructed $\mathbb{R}$-algebras with respective dimensions $\dim_{\mathbb{R}}(\mathbb{C}) = 2$, $\dim_{\mathbb{R}}(\mathbb{H}) = 4$, $\dim_{\mathbb{R}}(\mathbb{O}) = 8$.

A *quadratic form* on the $\mathbb{K}$-space V is a map $q\colon V \longrightarrow \mathbb{K}$ for which

$$q(ax) = a^2 q(x)$$

whenever $a \in \mathbb{K}$ and $x \in V$ and also the associated map $b\colon V \times V \longrightarrow K$ given by polarization

$$b(x,y) = q(x+y) - q(x) - q(y)$$

is a nondegenerate bilinear form. (See Appendix **??** for a brief discussion of quadratic and bilinear forms.)

The $\mathbb{R}$-algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ furnish examples of composition $\mathbb{R}$-algebras. A *composition algebra* is a $\mathbb{K}$-algebra A with multiplicative identity, admitting a nondegenerate quadratic form $\delta\colon A \longrightarrow \mathbb{K}$ that is multiplicative:

$$\delta(x)\delta(y) = \delta(xy),$$

for all $x, y \in A$. The codimension 1 subspace $1^{\perp}$ consists of the *pure imaginary* elements of A, and (in characteristic not 2) the conjugation map $\overline{a1 + b} = a1 - b$, for $b \in 1^{\perp}$, is an antiautomorphism of A whose fixed point subspace is $\mathbb{K}1$.

In the above $\mathbb{R}$-algebras the form δ is given by $\delta(x)1 = x\bar{x}$:

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix} = a\bar{a} + \bar{b}b \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In $\mathbb{O}$ specifically, for $a, b, c, d, e, f, g, h \in \mathbb{R}$, we find

$$\delta(((a,b)_{\mathbb{R}},(c,d)_{\mathbb{R}})_{\mathbb{C}},((e,f)_{\mathbb{R}},(g,h)_{\mathbb{R}})_{\mathbb{C}})_{\mathbb{H}} = \delta(a,b,c,d,e,f,g,h) = \\ = a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2\,.$$

Thus in $\mathbb{O}$ (and so its subalgebras $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$) all nonzero vectors have nonzero norm.

An immediate consequence of the composition law is that an invertible element of A must have nonzero norm. As $\delta(x)1 = x\bar{x}$ in composition algebras, the converse is also true. Therefore if 0 is the only element of the composition algebra A with norm 0, then all nonzero elements are invertible and A is a *division algebra*. Prime examples are the division composition $\mathbb{R}$-algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. The following remarkable theorem of Hurwitz shows that this situation is typical

(1.2). Theorem. (Hurwitz' Theorem) *If A is a composition algebra over $\mathbb{K}$, then $\dim_{\mathbb{K}}(A)$ is 1, 2, 4, or 8.* □□

If the composition $\mathbb{K}$-algebra A is not a division algebra, then it is called *split*. It turns out that a split composition algebra over $\mathbb{K}$ is uniquely determined up to isomorphism by its dimension. In dimension 1, the algebra is $\mathbb{K}$ itself, always a division algebra. In dimension 4, a split composition $\mathbb{K}$-algebra is always $\mathrm{Mat}_2(\mathbb{K})$ with $\delta = \det$, and the diagonal matrices provide a split subalgebra of dimension 2.

Composition algebras of dimension 8 are called *octonion algebras*. The original is the real division algebra $\mathbb{O}$ presented above and due to Graves (1843, unpublished) and Cayley (1845) [SpV00, p. 23].

A split octonion algebra $\mathbb{O}^{\mathrm{sp}}(\mathbb{K})$ over any field $\mathbb{K}$ is provided by *Zorn's vector matrices* [Zor31]

$$m = \begin{pmatrix} a & \vec{b} \\ \vec{c} & d \end{pmatrix}$$

with $a, d \in \mathbb{K}$ and $\vec{b}, \vec{c} \in \mathbb{K}^3$. Multiplication is given by

$$\begin{pmatrix} a & \vec{b} \\ \vec{c} & d \end{pmatrix}\begin{pmatrix} x & \vec{y} \\ \vec{z} & w \end{pmatrix} = \begin{pmatrix} ax+\vec{b}\cdot\vec{z} & a\vec{y}+w\vec{b} \\ x\vec{c}+d\vec{z} & \vec{c}\cdot\vec{y}+dw \end{pmatrix} + \begin{pmatrix} 0 & \vec{c}\times\vec{z} \\ -\vec{b}\times\vec{y} & 0 \end{pmatrix}$$

using the standard dot (inner) and cross (outer, exterior, vector) products of 3-vectors. The associated norm is

$$\delta(m) = ad - \vec{b}\cdot\vec{c}\,.$$

For any $\vec{v}$ with $\vec{v}\cdot\vec{v} = k \neq 0$ the subalgebra of all

$$m = \begin{pmatrix} a & b\vec{v} \\ ck^{-1}\vec{v} & d \end{pmatrix}$$

is a copy of the split quaternion algebra $\mathrm{Mat}_2(F)$ with norm the usual determinant.

Zorn (and others) gave a slightly different version of the vector matrices, replacing our entry $\vec{c}$ with its negative. This gives the more symmetrical norm form $\delta(m) = ad + \vec{b} \cdot \vec{c}$ but makes the connection with standard matrix multiplication and determinants less clear.

Extending coefficients in a composition algebra produces a composition algebra (although this is more than an exercise). For every composition $\mathbb{K}$-algebra O, there is an extension $\mathbb{E}$ of degree at most 2 over $\mathbb{K}$ with $\mathbb{E} \otimes_{\mathbb{K}} O$ a split composition $\mathbb{E}$-algebra. In particular every composition algebra over algebraically closed $\mathbb{E}$ is split and so unique up to isomorphism. The split algebra over $\mathbb{C}$ (for instance given by Zorn's vector matrices) has two isomorphism classes of $\mathbb{R}$-forms—the class of the split algebra $\mathbb{O}^{\mathrm{sp}}(\mathbb{R})$ and that of the Cayley-Graves division algebra $\mathbb{O}$.

1.3 Jordan algebras

As mentioned above, the basic models for associative algebras are the endomorphism algebras $\mathrm{End}_{\mathbb{K}}(V)$ for some $\mathbb{K}$-space V and the related matrix algebras $\mathrm{Mat}_n(\mathbb{K})$. While Jordan and Lie algebras both have abstract varietal definitions (given below for Jordan algebras and in the next section for Lie algebras), they are first seen in canonical models coming from $\mathrm{End}_{\mathbb{K}}(V)$.

We start with the observation that any pure tensor from $V \otimes V$ is the sum of its symmetric and skew-symmetric parts:

$$v \otimes w = \frac{1}{2}(v \otimes w + w \otimes v) + \frac{1}{2}(v \otimes w - w \otimes v).$$

In 1933 P. Jordan [JvNW34] initiated the study of the $\mathbb{K}$-algebra $A^+ = (A, \mu^+) = (A, \circ)$ that is the associative $\mathbb{K}$-algebra A equipped with the *Jordan product*

$$\mu^+(x \otimes y) = x \circ y = \frac{1}{2}(xy + yx)\,.$$

This requires, of course, that the characteristic of the field $\mathbb{K}$ not be 2. We could also consider the algebra without the factor of $\frac{1}{2}$, but we keep it for various reasons—in particular $x \circ x = \frac{1}{2}(xx + xx) = xx = x^2$ and $1 \circ x = \frac{1}{2}(1x + x1) = x$.

The model for all Jordan algebras is then $\mathrm{End}^+_{\mathbb{K}}(V)$, the vector space of all $\mathbb{K}$-endomorphisms of V equipped with the Jordan product.

Clearly the algebra $\mathrm{End}^+_{\mathbb{K}}(V)$ is commutative. Not so obvious is the fact that we also have the identity

$$(x \circ x) \circ (y \circ x) = ((x \circ x) \circ y) \circ x\,,$$

for all $x, y \in \mathrm{End}^+_{\mathbb{K}}(V)$. (Exercise.)

We are led to the general, varietal definition: the $\mathbb{K}$-algebra A is a *Jordan algebra* if it is commutative and satisfies the identical relation

$$x^2(yx) = (x^2y)x\,.$$

The canonical models are $\mathrm{End}^+_{\mathbb{K}}(V)$ and so also $\mathrm{Mat}^+_n(\mathbb{K})$ (in finite dimension).

Any subspace of $\mathrm{End}^+_{\mathbb{K}}(V)$ that is closed under the Jordan product is certainly a Jordan subalgebra. Especially if τ is an automorphism of $\mathrm{End}_{\mathbb{K}}(V)$, then its fixed-point-space is certainly closed under the Jordan product and so is a subalgebra. More subtly, if τ is an antiautomorphism of $\mathrm{End}_{\mathbb{K}}(V)$, then it induces an automorphism of $\mathrm{End}^+_{\mathbb{K}}(V)$ whose fixed points are again a Jordan subalgebra.

For instance, in the $\mathbb{K}$-algebra $\mathrm{Mat}_n(\mathbb{K})$ the transpose map is an antiautomorphism, so the symmetric matrices from $\mathrm{Mat}_n(\mathbb{K})$ form a Jordan subalgebra of $\mathrm{Mat}^+_n(\mathbb{K})$. More generally, if A is a $\mathbb{K}$-algebra with an antiautomorphism $a \mapsto \bar{a}$ fixing $\mathbb{K}$, then we can try the same trick with the $\mathbb{K}$-algebra $\mathrm{Mat}_n(A)$. The transpose-conjugate map

$$\bar{\tau}\colon (a_{ij}) \mapsto (\bar{a}_{ji})$$

is then an antiautomorphism of $\mathrm{Mat}_n(A)$ (Exercise), and so the associated fixed space of Hermitian matrices

$$\mathrm{H}_n(A) = \{\, M \in \mathrm{Mat}_n(A) \mid M = M^{\bar{\tau}} \,\}$$

is closed under the Jordan product

$$M \circ N = \frac{1}{2}(MN + NM)\,.$$

If A is associative then we have a Jordan algebra. Indeed this with $A = \mathbb{C}$ and $\mathbb{K} = \mathbb{R}$ was the original motivation for the physicist Jordan: in quantum mechanics the observables for the Hilbert space $\mathbb{C}^n$ are characterized by the hermitian matrices $\mathrm{H}_n(\mathbb{C})$, a set which is not closed under the standard matrix product but is a real Jordan algebra under the Jordan product.

When A is not associative, there is no reason to assume that this gives $\mathrm{H}_n(A)$ the structure of an (abstract) Jordan algebra. But if we choose A to be an octonion algebra over $\mathbb{K}$ and let $n \leq 3$, then this is in fact the case. (Recall that the alternative law is a weak version of the associative law, so this is not completely unreasonable.)

For the octonion $\mathbb{K}$-algebra O, the Jordan algebra $\mathrm{H}_3(O)$ is called an *Albert algebra*. Each matrix of $\mathrm{H}_3(O)$ has the shape

$$\begin{pmatrix} a & \alpha & \beta \\ \bar{\alpha} & b & \gamma \\ \bar{\beta} & \bar{\gamma} & c \end{pmatrix}$$

with $a, b, c \in \mathbb{K}$ (the fixed field of conjugation in O) and $\alpha, \beta, \gamma \in O$. Thus the $\mathbb{K}$-dimension of the Albert algebra $\mathrm{H}_3(O)$ is $3 + 3 \times 8 = 27$.

1.4 Lie algebras and linear representation

In the previous section we only discussed the symmetric part of the tensor decomposition displayed at the beginning of the section. But even at the time of Jordan, the corresponding skew part had been studied for years, starting with the Norwegian Sophus Lie and soon followed by Killing and Cartan (see [Bo01] and [Haw00]). If A is an associative algebra, then we define a skew algebra $A^- = (A, \mu^-) = (A, [\cdot,\cdot])$ by furnishing A with the multiplication

$$\mu^-(x \otimes y) = [x, y] = xy - yx\,.$$

(Note that the scaling factor $\frac{1}{2}$ does not appear.) The algebras A^- and in particular $\mathrm{End}^-_{\mathbb{K}}(V)$ and $\mathrm{Mat}^-_n(\mathbb{K})$ are the canonical models for Lie algebras over $\mathbb{K}$.

In a given category, a representation of an object M is loosely a morphism of M into one of the canonical examples from the category. So a *linear representation* of a group M is a homomorphism from M to some $\mathrm{GL}_{\mathbb{K}}(V)$. With this in mind, we will say that a *linear representation* of an associative algebra A, a Jordan algebra J, and a Lie algebra L (all over $\mathbb{K}$), respectively, is a $\mathbb{K}$-algebra homomorphism φ belonging to, respectively, some

$$\mathrm{Hom}_{\mathbb{K}\mathsf{Alg}}(A, \mathrm{End}_{\mathbb{K}}(V))\,, \quad \mathrm{Hom}_{\mathbb{K}\mathsf{Alg}}(J, \mathrm{End}^+_{\mathbb{K}}(V))\,, \quad \mathrm{Hom}_{\mathbb{K}\mathsf{Alg}}(L, \mathrm{End}^-_{\mathbb{K}}(V))\,,$$

which in the finite dimensional case can be viewed as

$$\mathrm{Hom}_{\mathbb{K}\mathsf{Alg}}(A, \mathrm{Mat}_n(\mathbb{K}))\,, \quad \mathrm{Hom}_{\mathbb{K}\mathsf{Alg}}(J, \mathrm{Mat}^+_n(\mathbb{K}))\,, \quad \mathrm{Hom}_{\mathbb{K}\mathsf{Alg}}(L, \mathrm{Mat}^-_n(\mathbb{K}))\,.$$

The corresponding image of φ is then a *linear associative algebra*, *linear Jordan algebra*, or *linear Lie algebra*, respectively. The representation is *faithful* if its kernel is 0. The underlying space V or $\mathbb{K}^n$ is then an *A-module* which *carries* the representation and upon which the algebra *acts*.

It turns out that in each of these categories, many of the important examples are linear. For instance

(1.3). PROPOSITION. *Every associative algebra with a multiplicative identity element is isomorphic to a linear associative algebra.*

PROOF. Let A be an associative algebra. For each $x \in A$, consider the map $\mathrm{ad}\colon A \longrightarrow \mathrm{End}_{\mathbb{K}}(A)$ of Lemma (1.1), given by $x \mapsto \mathrm{ad}_x$ where $\mathrm{ad}_x\, a = xa$ as before. That lemma states that ad is a vector space homomorphism.

Thus we need to check that multiplication is respected. But the associative identity

$$(xy)a = x(ya)$$

can be restated as

$$\mathrm{ad}_{xy}\, a = \mathrm{ad}_x\, \mathrm{ad}_y\, a\,,$$

for all $x, y, a \in A$. Hence $\mathrm{ad}_{xy} = \mathrm{ad}_x\, \mathrm{ad}_y$ as desired.

The kernel of ad consists of those x with $xa = 0$ for all $a \in A$. In particular, the kernel is trivial if A contains an identity element. □

It is clear from the proof that the multiplicative identity plays only a small role—the result should and does hold in greater generality. But for us the main message is that the adjoint map is a representation of every associative algebra. The proposition should be compared with Cayley's Theorem which proves that every group is (isomorphic to) a faithful permutation group by looking at the regular representation, which is the permutation version of adjoint action.

What about Jordan and Lie representation? Of course we still have not defined general Lie algebras, but we certainly want to include all the subalgebras of $\mathrm{End}^-_{\mathbb{K}}(V)$ and $\mathrm{Mat}^-_n(\mathbb{K})$.

As above, the multiplication map μ of an arbitrary Lie algebra $A = (A, [\cdot,\cdot])$ will be written as a bracket, in deference to the commutator product in an associative algebra:

$$\mu(x \otimes y) = [x, y]\,.$$

In the linear Lie algebras $\mathrm{End}^-_{\mathbb{K}}(V)$ and $\mathrm{Mat}^-_n(\mathbb{K})$ we always have

$$[x, x] = xx - xx = 0\,,$$

so we require that an abstract Lie algebra satisfy the *null identical relation*

$$[x, x] = 0\,.$$

This identity is not linear, but we may "linearize" it by setting $x = y + z$. We then find

$$0 = [y + z, y + z] = [y, y] + [y, z] + [z, y] + [z, z] = [y, z] + [z, y]\,,$$

giving as an immediate consequence the linear *skew identical relation*

$$[y, z] = -[z, y]\,.$$

If $\mathrm{char}\mathbb{K} \neq 2$, these two identities are equivalent. (This is typical of linearized identities: they are equivalent to the original except where neutralized by the characteristic.)

Our experience with groups and associative algebras tells us that having adjoint representations available is of great benefit, so we make an initial hopeful definition:

> *A* Lie algebra *is an algebra* $({}_{\mathbb{K}}L, [\cdot,\cdot])$ *in which all squares* $[x, x]$ *are* 0 *and for which the* $\mathbb{K}$*-homomorphism* $\mathrm{ad}\colon L \longrightarrow \mathrm{End}^-_{\mathbb{K}}(L)$ *is a representation of* L*.*

Are $\mathrm{End}^-_{\mathbb{K}}(V)$ and $\mathrm{Mat}^-_n(\mathbb{K})$ Lie algebras in this sense? Indeed they are:

$$\begin{aligned}
\mathrm{ad}_x\,\mathrm{ad}_y\,a &= \mathrm{ad}_x(ya - ay)\\
&= x(ya - ay) - (ya - ay)x\\
&= xya - xay - yax + ayx
\end{aligned}$$

hence

$$\begin{aligned}[\mathrm{ad}_x, \mathrm{ad}_y]a &= (\mathrm{ad}_x \mathrm{ad}_y - \mathrm{ad}_y \mathrm{ad}_x)a \\ &= (xya - xay - yax + ayx) - (yxa - yax - xay + axy) \\ &= (xya - axy) - (yxa - ayx) \\ &= [xy, a] - [yx, a] \\ &= [xy - yx, a] \\ &= \mathrm{ad}_{[x,y]}\, a\,.\end{aligned}$$

That is, $[\mathrm{ad}_x, \mathrm{ad}_y] = \mathrm{ad}_{[x,y]}$, as desired.

Let us now unravel the consequences of the identity $\mathrm{ad}_{[x,y]} = [\mathrm{ad}_x, \mathrm{ad}_y]$ for the algebra $(L, [\cdot,\cdot])$:

$$\begin{aligned}\mathrm{ad}_{[x,y]}\, z &= [\mathrm{ad}_x, \mathrm{ad}_y]z \\ [[x,y],z] &= (\mathrm{ad}_x \mathrm{ad}_y - \mathrm{ad}_y \mathrm{ad}_x)z \\ [[x,y],z] &= (\mathrm{ad}_x \mathrm{ad}_y)z - (\mathrm{ad}_y \mathrm{ad}_x)z \\ [[x,y],z] &= [x,[y,z]] - [y,[x,z]] \\ [[x,y],z] &= -[[y,z],x] - [[z,x],y]\,.\end{aligned}$$

That is,

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0\,.$$

We arrive at the standard definition of a Lie algebra:

A Lie algebra *is an algebra* $({}_{\mathbb{K}}L, [\cdot,\cdot])$ *that satisfies the two identical relations:*

(i) $[x,x] = 0$*;*

(ii) (Jacobi Identity) $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$.

Negating the Jacobi Identity gives us the equivalent identity

$$[z,[x,y]] + [x,[y,z]] + [y,[z,x]] = 0\,.$$

In particular, the opposite of a Lie algebra is again a Lie algebra.[1]

The Jacobi Identity and the skew law $[y,z] = -[z,y]$ are both linear, and these serve to define Lie algebras if the characteristic is not 2. This is good enough to prove that tensor product field extensions of Lie algebras are still Lie algebras as long as the characteristic is not 2

In all characteristics the null law $[x,x] = 0$ admits a weaker form of linearity. Assume that we already know $[y,y] = 0$, $[z,z] = 0$, and $[y,z] = -[z,y]$. Then for all constants a, b we have

$$\begin{aligned}[ay + bz, ay + bz] &= [ay, ay] + [ay, bz] + [bz, ay] + [bz, bz] \\ &= a^2[y,y] + ab([y,z] + [z,y]) + b^2[z,z] \\ &= 0 + 0 + 0 = 0\,.\end{aligned}$$

[1] Exercise: the map $x \mapsto -x$ is an isomorphism of the Lie algebra L with its opposite algebra.

This, together with the linearity of the Jacobi Identity, gives

(1.4). PROPOSITION. *Let L be Lie $\mathbb{K}$-algebra and $\mathbb{E}$ an extension field over $\mathbb{K}$. Then $\mathbb{E} \otimes_{\mathbb{K}} L$ is a Lie $\mathbb{E}$-algebra.* □

Our discussion of representation and our ultimate definition of Lie algebras immediately give

(1.5). THEOREM. *For any Lie $\mathbb{K}$-algebra L, the map $\mathrm{ad}\colon L \longrightarrow \mathrm{End}^-_{\mathbb{K}}(L)$ is a representation of L. The kernel of this representation is the* center *of L*

$$\mathrm{Z}(L) = \{\, z \in L \mid [z, a] = 0\,,\ \textit{for all } a \in L\,\}\,. \qquad \square$$

As was the case in Proposition (1.3) the small additional requirement that the center of A be trivial gives an easy proof that A has a faithful representation which has finite dimension provided A does. Far deeper is:

(1.6). THEOREM.

(a) (PBW THEOREM) *Every Lie algebra has a faithful representation as a linear Lie algebra.*

(b) (ADO-IWASAWA THEOREM) *Every finite dimensional Lie algebra has a faithful representation as a finite dimensional linear Lie algebra.* □□

Both these theorems are difficult to prove, although we will return to the easier PBW Theorem later as Theorem **??**. Notice that the Ado-Iwasawa Theorem is not an immediate consequence of PBW. Indeed the representation produced by the PBW Theorem is almost always a representation on an infinite dimensional space.

For Jordan algebras, the efforts of this section are largely a failure. In particular the adjoint action of a Jordan algebra A on itself does not give a representation in $\mathrm{End}^+_{\mathbb{K}}(A)$. (Exercise.)

Jordan algebras that are (isomorphic to) linear Jordan algebras are usually called *special Jordan algebras*, while those that are not linear are the *exceptional Jordan algebras*.[2] A.A. Albert [Alb34] proved that the Albert algebras—the dimension 27 Jordan $\mathbb{K}$-algebras described in Section 1.3—are exceptional rather than special. Indeed Cohn [Coh54] proved that Albert algebras are not even quotients of special algebras. Results of Birkhoff imply that the category of images of special Jordan algebras is varietal and does not contain the Albert algebras, but it is unknown what additional identical relations suffice to define this category.

1.5 Problems

(1.7). PROBLEM.

[2] So, taking a page out of the Montessori book, there are exactly two types of Jordan algebras: those that are special and those that are exceptional.

(a) *Give two linear identities that characterize alternative* $\mathbb{K}$*-algebras when* $\mathrm{char}\mathbb{K} \neq 2$.

(b) *Let* A *be an alternative* $\mathbb{K}$*-algebra and* $\mathbb{E}$ *an extension field over* $\mathbb{K}$*. Prove that* $\mathbb{E} \otimes_{\mathbb{K}} A$ *is an alternative* $\mathbb{E}$*-algebra.*

(1.8). PROBLEM. *Let* A *be an associative* $\mathbb{K}$*-algebra for* $\mathbb{K}$ *a field of characteristic not equal to* 2.

(a) *Prove that in general the adjoint action of a Jordan algebra does not give a representation. Consider specifically the Jordan algebra* $A^+ = (A, \circ)$ *and its adjoint map* $\mathrm{ad}\colon A^+ \longrightarrow \mathrm{End}^+_{\mathbb{K}}(A)$ *where you can compare* $\mathrm{ad}_{a\circ a}$ *and* $\mathrm{ad}_a \circ \mathrm{ad}_a$.

(b) *Consider the two families of maps from* A *to itself:*

$$L_a\colon x \mapsto a \circ x = \frac{1}{2}(ax + xa)$$

and

$$U_a\colon x \mapsto axa\,.$$

Prove that the $\mathbb{K}$*-subspace* V *of* A *is invariant under all* L_a*, for* $a \in V$*, if and only if it is invariant under all* U_a*, for* $a \in V$.

HINT: *The two parts of this problem are not unrelated.*

REMARK. *Observe that saying* V *is invariant under the* L_a *is just the statement that* V *is a Jordan subalgebra of* $\mathrm{End}^+_{\mathbb{K}}(A)$*, the map* L_a *being the adjoint. Therefore the problems tells us that requiring* U_a*-invariance is another way of locating Jordan subalgebras, for instance the important and motivating spaces of hermitian matrices* $\mathrm{H}_n(\mathbb{C})$ *in* $\mathrm{Mat}_n(\mathbb{C})$.

The crucial thing about U_a *is that division by* 2 *is not needed. Therefore the maps* U_a *and their properties can be, and are, used to extend the study of Jordan algebras to include characteristic* 2*. The appropriate structures are called* quadratic Jordan algebras*, although some care must be taken as the "multiplication"* $a \star x = U_a(x)$ *is not bilinear. It is linear in its second variable but quadratic in its first variable; for instance* $(\alpha a) \star x = \alpha^2 (a \star x)$ *for* $\alpha \in \mathbb{K}$.

Chapter 2

Examples of Lie algebras

We give many examples of Lie algebras $({}_{\mathbb{K}}L, [\cdot,\cdot])$. These also suggest the many contexts in which Lie algebras are to be found.

2.1 Abelian algebras

Any $\mathbb{K}$-vector space V is a Lie $\mathbb{K}$-algebra when provided with the trivial product $[v, w] = 0$ for all $v, w \in V$. These are the *abelian Lie algebras.*

2.2 Generators and relations

As with groups and most other algebraic systems, one effective way of producing examples is by providing a generating set and a collection of relations among the generators. For a $\mathbb{K}$-algebra that would often be through supplying a basis $\mathcal{V} = \{ v_i \mid i \in I \}$ together with appropriate equations restricting the various associated c_{ij}^k.

For a Lie algebra, the Jacobi Identity is linear and leads to (Exercise) the equations:

$$\sum_k c_{ij}^k c_{kl}^m + c_{jl}^k c_{kl}^m + c_{li}^k c_{kj}^m = 0\,,$$

for all $i, j, l, m \in I$.

The law $[x, x] = 0$ gives the equations

$$c_{ii}^k = 0\,.$$

Since the null law is not linear, we also must include the consequences of its linearized skew law $[x, y] = -[y, x]$; so we also require

$$c_{ij}^k = -c_{ji}^k\,.$$

An algebra whose multiplication coefficients satisfy these three sets of equations is a Lie algebra. (Exercise.)

When presenting a Lie algebra it is usual to leave the non-Jacobi equations implicit, assuming without remark that the bracket multiplication is null and skew-symmetric.

For instance, we have the $\mathbb{K}$-algebra $L = \mathbb{K}h \oplus \mathbb{K}e \oplus \mathbb{K}f$ where we state

$$[e, f] = h\,, \quad [h, e] = 2e\,, \quad [h, f] = -2f\,,$$

but in the future will not record the additional, necessary, but implied relations, which in this case are

$$[h, h] = [e, e] = [f, f] = 0\,, \quad [f, e] = -h\,, \quad [e, h] = -2e\,, \quad [f, h] = 2f\,.$$

Of course at this point in order to be sure that L really is a Lie algebra, we must verify the Jacobi Identity equations for all quadruples $(i, j, l, m) \in \{h, e, f\}^4$. (Exercise.)

2.3 Matrix algebras

2.3.1 Standard subalgebras of $\mathfrak{gl}_n(\mathbb{K})$

Many Lie algebras occur naturally as matrix algebras. We have already mentioned $\mathrm{Mat}_n^-(\mathbb{K})$. This is often written $\mathfrak{gl}_n(\mathbb{K})$, the *general linear algebra*, in part because it is the Lie algebra of the Lie group $\mathrm{GL}_n(\mathbb{K})$; see Theorem **??**(a) below. The Gothic (or Fraktur) font is also a standard for Lie algebras.

A standard matrix calculation shows that $\mathrm{tr}(MN) = \mathrm{tr}(NM)$, so the subset of matrices of trace 0 is a dimension $n^2 - 1$ subalgebra $\mathfrak{sl}_n(\mathbb{K})$ of the algebra $\mathfrak{gl}_n(\mathbb{K})$, which itself has dimension n^2. Indeed the *special linear algebra* $\mathfrak{sl}_n(\mathbb{K})$ is the *derived subalgebra* $[\mathfrak{gl}_n(\mathbb{K}), \mathfrak{gl}_n(\mathbb{K})]$ spanned by all $[M, N]$ for $M, N \in \mathfrak{gl}_n(\mathbb{K})$; see Section **??** below.

The subalgebras $\mathfrak{n}_n^+(\mathbb{K})$ and $\mathfrak{n}_n^-(\mathbb{K})$ are, respectively, composed of all strictly upper triangular and all strictly lower triangular matrices. Both have dimension $\binom{n}{2}$. Next let $\mathfrak{d}_n(\mathbb{K})$ and $\mathfrak{h}_n(\mathbb{K})$ be the abelian subalgebras of, respectively, all diagonal matrices (dimension n) and all diagonal matrices of trace 0 (dimension $n - 1$). We have the *triangular decomposition*

$$\mathfrak{gl}_n(\mathbb{K}) = \mathfrak{n}_n^+(\mathbb{K}) \oplus \mathfrak{d}_n(\mathbb{K}) \oplus \mathfrak{n}_n^-(\mathbb{K})$$

and

$$\mathfrak{sl}_n(\mathbb{K}) = \mathfrak{n}_n^+(\mathbb{K}) \oplus \mathfrak{h}_n(\mathbb{K}) \oplus \mathfrak{n}_n^-(\mathbb{K})\,.$$

This second decompositions and ones resembling it will be important later.

Within the Lie algebra $\mathfrak{sl}_2(\mathbb{K})$, consider the three elements

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

so that $\mathfrak{h}_2(\mathbb{K}) = \mathbb{K}h$, $\mathfrak{n}_2^+(\mathbb{K}) = \mathbb{K}e$, and $\mathfrak{n}_2^-(\mathbb{K}) = \mathbb{K}f$, and

$$\mathfrak{sl}_2(\mathbb{K}) = \mathbb{K}h \oplus \mathbb{K}e \oplus \mathbb{K}f\,.$$

We then have (Exercise)

$$[e, f] = h\,, \quad [h, e] = 2e\,, \quad [h, f] = -2f\,,$$

and the algebra presented at the end of the previous section is indeed a Lie algebra, namely a copy of $\mathfrak{sl}_2(\mathbb{K})$.

The isomorphism of $\mathrm{Mat}_n(\mathbb{K})$ and $\mathrm{End}_{\mathbb{K}}(\mathbb{K}^n)$ lead to natural isomorphisms of the above subalgebras of $\mathfrak{gl}_n(\mathbb{K}) = \mathrm{Mat}_n^-(\mathbb{K})$ with subalgebras of $\mathrm{End}_{\mathbb{K}}^-(\mathbb{K}^n)$.

2.3.2 Lie algebras from forms

For the basic theory of bilinear forms, see Appendix **??**. For bilinear b, the $\mathbb{K}$-space of endomorphisms

$$\mathfrak{L}(V, b) = \{\, x \in \mathrm{End}_{\mathbb{K}}(V) \mid b(xv, w) = -b(v, xw) \text{ for all } v, w \in V \,\}$$

is then an Lie $\mathbb{K}$-subalgebra of $\mathrm{End}_{\mathbb{K}}^-(V)$. (Exercise.)

With $V = \mathbb{K}^n$ and $\mathrm{End}_{\mathbb{K}}(V) = \mathrm{Mat}_n(\mathbb{K}) = \mathfrak{gl}_n(K)$, we have some special cases of $\mathfrak{L} = \mathfrak{L}(V, b)$. Let $G = (b(e_i, e_j))_{i,j}$ be the *Gram matrix* of b on V (with respect to the usual basis). The condition above then becomes

$$\mathfrak{L}(V, b) = \{\, M \in \mathrm{Mat}_n(\mathbb{K}) \mid MG = -GM^\top \,\}\,.$$

For simplicity's sake we assume that $\mathbb{K}$ does not have characteristic 2.

(i) **Orthogonal algebras.**

(a) If b is the usual nondegenerate orthogonal form with an orthonormal basis, then $\mathfrak{L} = \mathfrak{so}_n(\mathbb{K})$. As matrices,

$$\mathfrak{so}_n(\mathbb{K}) = \{\, M \in \mathrm{Mat}_n(\mathbb{K}) \mid M = -M^\top \,\}\,.$$

(b) If the field $\mathbb{K}$ is algebraically closed, then it is always possible to find a basis for which the Gram matrix G is in split form as the $2l \times 2l$ matrix with l blocks $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ down the diagonal when $n = 2l$ is even, and this same matrix with an additional single 1 on the diagonal when $n = 2l + 1$ is odd.

For the split form over an arbitrary field $\mathbb{K}$, we may write $\mathfrak{so}_{2l}^+(\mathbb{K})$ in place of $\mathfrak{so}_{2l}(\mathbb{K})$.

(ii) **Symplectic algebras.** If b is the usual nondegenerate (split) symplectic form with symplectic basis $\mathcal{S} = \{\, v_i, w_i \mid 1 \le i \le l \,\}$ subject to $b(v_i, v_j) = b(w_i, w_j) = 0$ and $b(v_i, w_j) = \delta_{i,j} = -b(w_j, v_i)$, then $\mathfrak{sp}_{2l}(\mathbb{K}) = \mathfrak{L}$. As matrices,

$$\mathfrak{sp}_{2l}(\mathbb{K}) = \{\, M \in \mathrm{Mat}_{2l}(\mathbb{K}) \mid MG = -GM^\top \,\}\,,$$

where G is the $2l \times 2l$ matrix with n blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ down the diagonal.

The notation is not uniform. Especially, when $\mathbb{K} = \mathbb{R}$ the field is sometimes omitted, hence one may find

$$\mathfrak{gl}_n(\mathbb{R}) = \mathfrak{gl}(n, \mathbb{R}) = \mathfrak{gl}(n) = \mathfrak{gl}_n\,, \quad \mathfrak{sl}_n(\mathbb{R}) = \mathfrak{sl}(n, \mathbb{R}) = \mathfrak{sl}(n) = \mathfrak{sl}_n\,;$$

and

$$\mathfrak{so}_n(\mathbb{R}) = \mathfrak{so}(n, \mathbb{R}) = \mathfrak{so}(n) = \mathfrak{so}_n\,.$$

More confusingly, in the case of symplectic algebras the actual definition can vary as well as the notation; see [Tu11, p. 160].

2.4 Derivations

A *derivation* D on the $\mathbb{K}$-algebra A is a linear transformation $D \in \mathrm{End}_{\mathbb{K}}(A)$ with

$$D(fg) = fD(g) + D(f)g\,,$$

for all $f, g \in A$. This should be recognized as the *Leibniz product rule*. Clearly the set $\mathrm{Der}_{\mathbb{K}}(A)$ is a $\mathbb{K}$-subspace of $\mathrm{End}_{\mathbb{K}}(A)$, but in fact this provides an amazing machine for constructing Lie algebras:

(2.1). THEOREM. $\mathrm{Der}_{\mathbb{K}}(A) \leq \mathrm{End}^-_{\mathbb{K}}(A)$. *That is, the derivation space is a Lie* $\mathbb{K}$-*algebra under the bracket product.*

PROOF. Let $D, E \in \mathrm{Der}_{\mathbb{K}}(A)$. Then, for all $f, g \in A$,

$$\begin{aligned}
[D,E](fg) &= (DE - ED)(fg) = DE(fg) - ED(fg)\\
&= D(fEg + (Ef)g) - E(fDg + (Df)g)\\
&= D(fEg) + D((Ef)g) - E(fDg) - E((Df)g))\\
&= fDEg + DfEg + EfDg + (DEf)g\\
&\quad - fEDg - EfDg - DfEg - (EDf)g\\
&= fDEg - fEDg + (DEf)g - (EDf)g\\
&= f([D,E]g) + ([D,E]f)g\,. \qquad \square
\end{aligned}$$

The definition of derivations then tells us that the injection of $\mathrm{Der}_{\mathbb{K}}(A)$ into $\mathrm{End}^-_{\mathbb{K}}(A)$ gives a representation of the Lie *derivation algebra* $\mathrm{Der}_{\mathbb{K}}(A)$ on the $\mathbb{K}$-space A.

(2.2). COROLLARY. *The image of the Lie algebra A under the adjoint representation is a subalgebra of* $\mathrm{Der}_{\mathbb{K}}(A)$ *and* $\mathrm{End}^-_{\mathbb{K}}(A)$.

PROOF. The image of A under ad is a $\mathbb{K}$-subspace of $\mathrm{End}_{\mathbb{K}}(A)$ by our very first Lemma (1.1). It remains to check that each ad_a is a derivation of A.

We start from the Jacobi Identity:

$$[[a, y], z] + [[y, z], a] + [[z, a], y] = 0\,,$$

hence

$$-[[y,z],a] = [[a,y],z] + [[z,a],y]\,.$$

That is,

$$[a,[y,z]] = [[a,y],z] + [y,[a,z]]\,,$$

or

$$\mathrm{ad}_a[y,z] = [\mathrm{ad}_a\, y, z] + [y, \mathrm{ad}_a\, z]\,. \qquad \square$$

The map ad_a is then an *inner derivation* of A, and the Lie subalgebra $\mathrm{InnDer}_{\mathbb{K}}(A) = \{\,\mathrm{ad}_a \mid a \in A\,\}$ is the *inner derivation algebra.*

We have an easy but useful observation:

(2.3). PROPOSITION. *Every linear transformation of* $\mathrm{End}_{\mathbb{K}}(A)$ *is a derivation of the abelian Lie algebra* A*.*

PROOF. For $D \in \mathrm{End}_{\mathbb{K}}(A)$ and $a, b \in A$

$$D[a,b] = 0 = 0 + 0 = [Da, b] + [a, Db]\,. \qquad \square$$

2.4.1 Derivations of polynomial algebras

(2.4). PROPOSITION.

(a) $\mathrm{Der}_{\mathbb{K}}(\mathbb{K}) = 0$.

(b) *If the* $\mathbb{K}$*-algebra* A *has an identity element* 1*, then for each* $D \in \mathrm{Der}_{\mathbb{K}}(A)$ *and each* $c \in \mathbb{K}1$ *we have* $D(c) = 0$*.*

(c) $\mathrm{Der}_{\mathbb{K}}(\mathbb{K}[t]) = \{\, p(t)\frac{d}{dt} \mid p(t) \in \mathbb{K}[t]\,\}$*, a Lie algebra of infinite* $\mathbb{K}$*-dimension with basis* $\{\, t^i \frac{d}{dt} \mid i \in \mathbb{N}\,\}$*.*

PROOF. Part (b) clearly implies (a).

(b) Let $c = c1 \in \mathbb{K}1$. Then for all $x \in A$ and all $D \in \mathrm{Der}_{\mathbb{K}}(A)$ we have

$$D(cx) = cD(x)$$

as D is a $\mathbb{K}$-linear transformation. But D is also a derivation, so

$$D(cx) = cD(x) + D(c)x\,.$$

We conclude that $D(c)x = 0$ for all $x \in A$, and so $D(c) = 0$.

(c) Let $D \in \mathrm{Der}_{\mathbb{K}}(A)$. By (b) we have $D(\mathbb{K}1) = 0$. As the algebra A is generated by 1 and t, the knowledge of $D(t)$ together with the product rule should give us everything. Set $p(t) = D(t)$.

We claim that $D(t^i) = p(t)it^{i-1}$ for all $i \in \mathbb{N}$. We prove this by induction on i, the result being clear for $i = 0, 1$. Assume the claim for $i - 1$. Then

$$\begin{aligned} D(t^i) &= D(t^{i-1}t) = t^{i-1}D(t) + D(t^{i-1})t \\ &= t^{i-1}p(t) + p(t)(i-1)t^{i-2}t = p(t)it^{i-1}\,, \end{aligned}$$

as claimed.

As D is a linear transformation, if $a(t) = \sum_{i=0}^{m} a_i t^i$, then

$$D(a(t)) = \sum_{i=0}^{m} a_i D(t^i) = \sum_{i=0}^{m} a_i p(t) i t^{i-1} = p(t) \sum_{i=0}^{m} i a_i t^{i-1} = p(t) \frac{d}{dt} a(t) ,$$

completing the proposition. □

In $\mathrm{Der}_{\mathbb{K}}(\mathbb{K}[t])$ there is the subalgebra $A = \mathbb{K}h \oplus \mathbb{K}e \oplus \mathbb{K}f$ with $e = \frac{d}{dt}$, $h = -2t\frac{d}{dt}$, $f = -t^2\frac{d}{dt}$, and relations (Exercise)

$$[e, f] = h , \quad [h, e] = 2e , \quad [h, f] = -2f ;$$

so we have $\mathfrak{sl}_2(\mathbb{K})$ again.

We next consider $\mathbb{K}[x, y]$. A similar argument to that of the proposition proves

$$\mathrm{Der}_{\mathbb{K}}(\mathbb{K}[x, y]) = \{\, p(x, y)\frac{\partial}{\partial x} + q(x, y)\frac{\partial}{\partial y} \mid p(x, y), q(x, y) \in \mathbb{K}[x, y] \,\} .$$

(See Problem (2.8).) We examine two special situations—a subalgebra and a quotient algebra.

(i) Consider the Lie subalgebra that leaves each homogeneous piece of $\mathbb{K}[x, y]$ invariant. This subalgebra has basis

$$h_x = x\frac{\partial}{\partial x} , \; e = x\frac{\partial}{\partial y} , \; f = y\frac{\partial}{\partial x} , \; h_y = y\frac{\partial}{\partial y} .$$

Set $h = h_x - h_y = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$. Then

$$[e, f] = h , \quad [h, e] = 2e , \quad [h, f] = -2f ,$$

giving $\mathfrak{sl}_2(\mathbb{K})$ yet again. The 4-dimensional algebra $\mathbb{K}h_x \oplus \mathbb{K}h_y \oplus \mathbb{K}e \oplus \mathbb{K}f$ is isomorphic to $\mathfrak{gl}_2(\mathbb{K})$ with the correspondences

$$h_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad h_y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} .$$

Each homogeneous piece of $\mathbb{K}[x, y]$ carries a representation of $\mathfrak{gl}_2(\mathbb{K})$ and $\mathfrak{sl}_2(\mathbb{K})$ via restriction from the action of $\mathrm{Der}_{\mathbb{K}}(\mathbb{K}[x, y])$. The degree m homogeneous component $\mathbb{K}[x, y]_m$ is then a cyclic $\mathbb{K}e$- hence $\mathfrak{sl}_2(\mathbb{K})$-module $M_0(m+1)$ of dimension $m+1$ with generator y^m. This will be important in Chapter **??**.

(ii) The algebra $\mathbb{K}[x, y]$ has as quotient the algebra $\mathbb{K}[x, x^{-1}]$ of all *Laurent polynomials* in x. A small extension of the arguments from Proposition

(2.4)(c) (Exercise) proves that $\mathrm{Der}_{\mathbb{K}}(\mathbb{K}[x, x^{-1}])$ has $\mathbb{K}$-basis consisting of the distinct elements

$$L_m = -x^{m+1}\frac{d}{dx} \quad \text{for } m \in \mathbb{Z}\,.$$

We write the generators in this form, since they then have the nice presentation

$$[L_m, L_n] = (m-n)L_{m+n}\,.$$

All the multiplication coefficients are integers. The $\mathbb{Z}$-algebra with this presentation has infinite dimension. It is called the *Witt algebra* over $\mathbb{Z}$, just as its tensor with $\mathbb{K}$, $\mathrm{Der}_{\mathbb{K}}(\mathbb{K}[x, x^{-1}])$, is the Witt algebra over $\mathbb{K}$.

2.4.2 Derivations of nonassociative algebras

We may also consider derivations of the nonassociative algebras we have encountered, specifically the octonion $\mathbb{K}$-algebra O and (in characteristic not 2) its related Albert algebra—the exceptional Jordan $\mathbb{K}$-algebra $\mathrm{H}_3(O)$. The derivation algebra $\mathrm{Der}_{\mathbb{K}}(O)$ has dimension 14 (when $\mathrm{char}\mathbb{K} \neq 3$) and is said to have *type* $\mathfrak{g}_2$ while the algebra of inner derivations of the Albert algebra $\mathrm{H}_3(O)$ has dimension 52 and is said to have *type* $\mathfrak{f}_4$. Especially when $\mathbb{K}$ is algebraically closed and of characteristic 0 we have the uniquely determined algebras $\mathfrak{g}_2(\mathbb{K})$ and $\mathfrak{f}_4(\mathbb{K})$, respectively.

2.5 New algebras from old

2.5.1 Extensions

As we have seen and expect, subalgebras and quotients are ways of constructing new algebras out of old algebras. We can also extend old algebras to get new ones. As with groups, central extensions are important since the information we have about a given situation may come to us, via the adjoint, in projective rather than affine form.

The *Virasoro algebra* is a central extension of the complex Witt algebra. If W is the Witt $\mathbb{Z}$-algebra, then

$$\mathrm{Vir}_{\mathbb{C}} = (\mathbb{C} \otimes_{\mathbb{Z}} W) \oplus \mathbb{C}c$$

with $[w, c] = 0$ for all $w \in W$ and

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m(m^2-1)}{12}c\,.$$

The multiplication coefficients are half-integers.

The Virasoro algebra is important in applications to physics and other situations. As seen after Proposition (2.4), the Witt and Virasoro algebras both contain the subalgebra $\mathbb{C}L_{-1} \oplus \mathbb{C}L_0 \oplus \mathbb{C}L_1$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. As we shall find starting in Section **??**, large parts of the finite dimensional Lie algebra theory depend upon the construction of Lie subalgebras $\mathfrak{sl}_2(\mathbb{K})$. Similarly, the infinite dimensional Lie algebras that come up in physics and elsewhere are often handled using Witt and Virasoro subalgebras, which are in a sense the infinite dimensional substitutes for the finite dimensional $\mathfrak{sl}_2(\mathbb{K})$.

Given a complex simple Lie algebra like $\mathfrak{sl}_2(\mathbb{C})$, the corresponding *affine Lie algebra* comes from a two step process. First extend scalars to the Laurent polynomials and second take an appropriate central extension. So:

$$\widehat{\mathfrak{sl}}_2(\mathbb{C}) = (\mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}} \mathfrak{sl}_2(\mathbb{C})) \oplus \mathbb{C}c$$

where the precise cocycle on the complex Lie algebra $\mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}} \mathfrak{sl}_2(\mathbb{C})$ that gives the extension is defined in terms of the Killing form on the algebra $\mathfrak{sl}_2(\mathbb{C})$. (See Chapter **??** below.)

One often writes the Lie algebra $\mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}} \mathfrak{sl}_2(\mathbb{C})$ instead as $\mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[t,t^{-1}]$, viewing its elements as "Laurent polynomials" with coefficients from the algebra $\mathfrak{sl}_2(\mathbb{C})$.

It is also possible to form split extensions of Lie algebras, with derivations playing the role that automorphisms play in group extensions. (See Section **??**.) The canonical derivation $\frac{d}{dt}$ on the Laurent polynomials induces a derivation of the affine algebra which is then used to extend the affine algebra so that it has codimension 1 in the corresponding *Kac-Moody Lie algebra*.

2.5.2 Embeddings

We saw above that derivations of octonion and Jordan algebras give new Lie algebras. Tits, Kantor, and Koecher [Tit66] used these same nonassociative algebras to construct (the *TKK construction*) Lie algebras that are still more complicated. In particular, the space

$$\mathrm{Der}_{\mathbb{C}}(\mathbb{O}^{\mathrm{sp}}(\mathbb{C})) \;\oplus\; (\mathbb{O}^{\mathrm{sp}}(\mathbb{C})_0 \otimes_{\mathbb{C}} \mathrm{H}_3(\mathbb{O}^{\mathrm{sp}}(\mathbb{C}))_0) \;\oplus\; \mathrm{Der}_{\mathbb{C}}(\mathrm{H}_3(\mathbb{O}^{\mathrm{sp}}(\mathbb{C})))$$

of dimension $14+(8-1)\times(27-1)+52 = 248$ can be provided with a Lie algebra product (extending that of the two derivation algebra pieces) that makes it into the Lie algebra $\mathfrak{e}_8(\mathbb{C})$. Here $\mathbb{O}^{\mathrm{sp}}(\mathbb{C})_0$ is $1^\perp$ in $\mathbb{O}^{\mathrm{sp}}(\mathbb{C})$ and $\mathrm{H}_3(\mathbb{O}^{\mathrm{sp}}(\mathbb{C}))_0$ is a similarly defined subspace of codimension 1 in $\mathrm{H}_3(\mathbb{O}^{\mathrm{sp}}(\mathbb{C}))$. The Lie algebra $\mathfrak{e}_8(\mathbb{C})$ furthermore has the important subalgebras $\mathfrak{e}_6(\mathbb{C})$ of dimension 78 and $\mathfrak{e}_7(\mathbb{C})$ of dimension 133.

2.6 Other contexts

2.6.1 Nilpotent groups

Let G be a nilpotent group with *lower central series*

$$G = \mathrm{L}^1(G) \trianglerighteq \mathrm{L}^2(G) \trianglerighteq \cdots \trianglerighteq \mathrm{L}^{n+1}(G) = 1$$

where $\mathrm{L}^{k+1}(G)$ is defined as $[G, \mathrm{L}^k(G)]$. For each $1 \leq k \leq n$ set

$$L_k = \mathrm{L}^k(G)/\mathrm{L}^{k+1}(G)\,,$$

an abelian group as is the sum

$$L = \bigoplus_{k=1}^{n} L_k\,.$$

As G is nilpotent, always

$$[\mathrm{L}^i(G), \mathrm{L}^j(G)] \leq \mathrm{L}^{i+j}(G)\,.$$

This provides the relations that turn the group $L = L_G$ into a *Lie ring*—we do not require it to be free as $\mathbb{Z}$-module—within which we have

$$[L_i, L_j] \leq L_{i+j}\,.$$

Certain questions about nilpotent groups are much more amenable to study in the context of Lie rings and algebras [Hig58]. A particular important instance is the *Restricted Burnside Problem*, which states that an m-generated finite nilpotent group of exponent e has order less than or equal to some function $f(m, e)$, dependent only on m and e. Professor E. Zelmanov received a Fields Medal in 1994 for the positive solution of the Restricted Burnside Problem. His proof [Zel97] makes heavy use of Lie methods.

2.6.2 Vector fields

We shall see in the next chapter that the tangent space to a Lie group at the identity is a Lie algebra. As the group acts regularly on itself by translation, this space is isomorphic to the Lie algebra of invariant vector fields on the group.

Indeed often a *vector field* on the smooth manifold M is defined to be a derivation of the algebra $\mathrm{C}^\infty(M)$ of all smooth functions; for instance, see [Hel01, p. 9]. Thus the space of all vector fields is the corresponding derivation algebra and so automatically has a Lie algebra structure.

For instance, the Lie group of rotations of the circle S^1 is the group $\mathrm{SO}_2(\mathbb{R})$ of all matrices

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},$$

which becomes $e^{i\theta}$ when we extend coefficients to the complex numbers. The corresponding spaces of invariant vector fields have dimension 1.

The space $\mathrm{C}^\infty(\mathrm{S}^1)$ of all smooth functions on the circle consists of those functions that can be expanded as convergent *Fourier series*

$$\sum_{m\in\mathbb{Z}} a_m \sin(m\theta) + b_m \cos(m\theta)\,,$$

which after extension to $\mathbb{C}$ becomes the simpler

$$\sum_{m\in\mathbb{Z}} c_m e^{im\theta}\,.$$

This space has as a dense subalgebra the space of all *Fourier polynomials*, whose canonical basis is $\{\, e^{im\theta} \mid m \in \mathbb{Z}\,\}$.

The group of all complex orientation preserving diffeomorphisms of the circle (an "infinite dimensional Lie group") is an open subset of $\mathrm{C}^\infty_{\mathbb{C}}(\mathrm{S}^1)$ and has as corresponding space of smooth vector fields (not just those that are invariant) all $f\frac{d}{d\theta}$ for f smooth. The dense Fourier polynomial subalgebra with basis $L_m = ie^{im\theta}\frac{d}{d\theta}$ then has

$$[L_m, L_n] = (m-n)L_{m+n}\,,$$

giving the complex Witt algebra again.

2.7 Problems

(2.5). PROBLEM. *Classify up to isomorphism all Lie $\mathbb{K}$-algebras of dimension 2. (Of course, the abelian algebra gives the only isomorphism class in dimension 1.)*

(2.6). PROBLEM. *Prove that over an algebraically closed field $\mathbb{K}$ of characteristic not 2, the Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ is isomorphic to $\mathfrak{so}_3(\mathbb{K})$, the orthogonal Lie algebra of 3×3 skew-symmetric matrices.*

(2.7). PROBLEM. *Find all subalgebras of $\mathfrak{sl}_2(\mathbb{K})$ that contain the subalgebra $H = \mathbb{K}h$.* HINT: *Small characteristic can produce anomalous results.*

(2.8). PROBLEM. *Calculate $\mathrm{Der}_{\mathbb{K}}(\mathbb{K}[x_1, \dots x_n])$.*

(2.9). PROBLEM. *Consider the matrix subgroup $\mathrm{UT}_n(\mathbb{K})$ of $\mathrm{GL}_n(\mathbb{K})$, consisting of the upper unitriangular matrices—those which have 1's on the diagonal, anything above the diagonal, and 0's below the diagonal.*

(a) *Prove that $G = \mathrm{UT}_n(\mathbb{K})$ is a nilpotent group.*

(b) *Starting with this group G, construct the Lie algebra $L = L_G$ as in Section 2.6.1. Prove that L is isomorphic to the Lie algebra $\mathfrak{n}^+_n(\mathbb{K})$.*

(2.10). PROBLEM. *Consider the subgroup $\mathrm{X}_n(\mathbb{K})$ of upper unitriangular matrices that have 1's on the diagonal, anything in the nondiagonal part of the first row and last column, and 0's elsewhere.*

(a) *By the previous problem $X = \mathrm{X}_n(\mathbb{K})$ is nilpotent. Prove that for $n \geq 2$ it has nilpotence class exactly 2 and that its center is equal to its derived group and consists only of those matrices with 1's down the diagonal and the only other nonzero entries found in the upper-righthand corner.*

(b) *Starting with this group X, construct the Lie algebra $L = L_X$ as in Section 2.6.1. Prove that L is isomorphic to the Lie algebra on the space*

$$M = \mathbb{K}z \oplus \bigoplus_{i=1}^{n-1} (\mathbb{K}x_i \oplus \mathbb{K}y_i)$$

with relations given by

$$[x_i, y_i] = -[y_i, x_i] = z\,,$$

for all i*, and all other brackets among generators equal to* 0*.*

REMARK. *This Lie algebra is the* Heisenberg algebra *of dimension* $2n-1$ *over* $\mathbb{K}$*.*

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