RESEARCH STATEMENT

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The primary focus of my recent work has been the study of algebraic cycles using Hodge theory. Section 1 contains a rough sketch of Hodge theory and an informal description of my results. Section 2 contains a more detailed account of my work assuming a basic knowledge of Hodge theory and algebraic geometry.

1. Introduction

In simplest terms, a differentiable manifold \( M \) is just a topological space upon which one has a notion of what it means for a continuous function \( f : M \to \mathbb{R} \) to be smooth, i.e. infinitely differentiable. Likewise, a complex manifold can be viewed as just a differentiable manifold upon which one has a notion of what it means for a smooth function to be holomorphic.

One of the basic results in differential topology (Whitney’s theorem) then asserts that every differentiable manifold \( M \) can be properly embedded as a submanifold of a Euclidean space \( \mathbb{R}^n \) of sufficiently high dimension. In contrast, a complex manifold \( M \) (of positive dimension) which is compact as a topological space can never be embedded as a complex submanifold of a complex Euclidean space \( \mathbb{C}^n \). Fortunately however, under suitable conditions, a compact complex manifold can be properly embedded as a complex submanifold of a sufficiently high dimensional projective space \( \mathbb{C}P^n \). Such manifolds are called smooth projective varieties, and can always be presented as the zero locus of a system of polynomial equations.

In its simplest form, Hodge theory is then the study of period integrals, i.e. integrals of algebraic differential forms on smooth projective varieties and their generalizations. For example, in the case of the elliptic curve defined by the equation

\[
w^2 = z(z - 1)(z - \lambda), \quad \lambda \in \mathbb{C} - \{0, 1\}
\]

the period integrals of interest are the elliptic integrals

\[
\pi_1(\lambda) = \int_0^1 \frac{dz}{\sqrt{z(z - 1)(z - \lambda)}}, \quad \pi_2(\lambda) = \int_1^\lambda \frac{dz}{\sqrt{z(z - 1)(z - \lambda)}}
\]

Using the Hodge decomposition theorem for Kähler manifolds, the period integrals on a smooth complex projective variety \( X \) can be packaged together to give a pure Hodge structure

\[
H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \quad H^{p,q}(X) = \bigoplus_{p+q=k} H_{p,q}(X)
\]

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of weight $k$ on the cohomology group $H^k(X)$.

In general, period integrals are transcendental numbers, and one of the basic themes of the subject is that what algebraic structure such period integrals do have is governed by algebraic cycles, i.e. formal linear sums of algebraic subvarieties. In particular, the Hodge conjecture asserts that every rational cohomology class of type $(p, p)$ on a smooth projective variety can be represented via an algebraic cycle.

Instead of considering a fixed variety $X$, let us now consider a family of smooth projective varieties $f: X \to S$. Then, by the work of Phillip Griffiths the pure Hodge structures $H^k(X_s)$ of weight $k$ on the fibers of $f$ determine an associated period map

$$\pi: S \to \Gamma \backslash D$$

where $\Gamma$ is a discrete group, $D$ classifies all pure Hodge structures of a given type and $\pi$ satisfies a differential constraint known as Griffiths’ horizontality. In the language of vector bundles, period maps correspond to objects called variations of Hodge structure.

To continue the story, let us now replace ordinary algebraic integrals by iterated integrals. Then, instead of being led to the theory of pure Hodge structures considered by Griffiths, one lands in the category of mixed Hodge structures constructed by Deligne. In this form, the subject can be said to date back to the classical problem of trying to analytically continue the series

$$L_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$$

from a single valued holomorphic function on the unit disk to a multivalued holomorphic function on $\mathbb{C} \setminus \{0, 1\}$. In particular, as was noted by Euler and his successors, this problem is equivalent to evaluating an iterated integral, since

$$L_k(z) = \int_0^z L_{k-1}(w) \frac{dw}{w}$$

The modern history of the subject begins with the work of Deligne who was able to show that the cohomology of an arbitrary algebraic variety $X$ defined over $\mathbb{C}$ (smooth, singular, etc.) admits a functorial mixed Hodge structure. Furthermore, to each family of algebraic varieties $f: X \to S$ one may attach a corresponding (mixed) period map $\pi: S^* \to \Gamma \backslash M$ defined over an open subset $S^* \subseteq S$ which satisfies a version of Griffiths’ horizontality. The corresponding vector bundle objects are called variations of mixed Hodge structure.

In an analogy with iterated integrals, a mixed Hodge structure can be thought of as a kind of iterated extension of pure Hodge structure. In particular, aside from pure Hodge structures, the simplest mixed Hodge structures are just extensions

$$0 \to A \to B \to C \to 0$$
where $A$ and $C$ are pure Hodge structures such that the weight of $C$ is strictly greater than the weight of $A$. A simple argument shows that

$$\text{Ext}^1_{\text{MHS}}(C, A) \cong \text{Ext}^1_{\text{MHS}}(\mathbb{Z}(0), H)$$

where $\mathbb{Z}(0)$ is a pure Hodge structure of dimension 1 and type $(0,0)$ and $H = A^* \otimes C$ is a pure Hodge structure of negative weight, i.e.

$$H_C = \bigoplus_{p+q=w} H^{p,q}, \quad \bar{H}^{p,q} = H^{q,p}$$

for some fixed integer $w < 0$.

In particular, when $H$ is pure of weight $-1$, the intermediate Jacobian

$$J(H) = \text{Ext}^1_{\text{MHS}}(\mathbb{Z}(0), H)$$

is a compact complex torus. Likewise, a variation of Hodge structure $H \to S$ of weight $-1$ defines a bundle of compact, complex tori $J(H) \to S$. A holomorphic section $\nu : S \to J(H)$ is then called a normal function if there exists an extension

$$0 \to H \to V \to \mathbb{Z}(0) \to 0$$

in the category of variations of mixed Hodge structures which induces the function $\nu$, i.e.

$$\nu(s) = [V_s] \in \text{Ext}^1_{\text{MHS}}(\mathbb{Z}(0), H_s) = J(H_s)$$

for each point $s \in S$.

As a holomorphic section of a torus bundle, the zero locus of $\nu$ is a complex analytic subvariety of $S$. One of my basic results [BP1, BP2, BP3] is that when $S$ is algebraic and $\nu$ satisfies a technical condition known as admissibility (which is true for all variations of geometric origin), the zero locus of $\nu$ is in fact an algebraic subvariety of $S$. Furthermore, as I explain in the next section, the existence of certain kinds of singularities of normal functions is equivalent to the Hodge conjecture.

\section{Normal Functions}

In [GG], Phillip Griffiths and Mark Green suggested the following method to prove the Hodge conjecture by induction on dimension using normal functions: Let $(X, L)$ be a pair consisting of a smooth complex projective variety $X$ of dimension $2n$ and a very ample line bundle $L \to X$. This gives an incidence variety

$$\mathcal{X} = \{ (x, p) \in X \times \tilde{P} \mid p(x) = 0 \}$$

where $\tilde{P} = |L|$. Let $\tilde{X} \subseteq \tilde{P}$ denote the dual variety of singular hyperplane sections and $P = \tilde{P} - \tilde{X}$. Let $\pi : \mathcal{X} \to \tilde{P}$ denote the second factor, and $\mathcal{H}$ be the variation of pure Hodge structure of weight $-1$ over $P$ associated to $R^{2n-1}_{\mathcal{X}*} \mathbb{Z}(n)$. 

Given a primitive integral Hodge class \( \zeta \) of type \((n,n)\) on \( X \), a choice of lifting of \( \zeta \) to Deligne cohomology determines an admissible normal function \( \nu_{\zeta,L} : P \to J(H) \) i.e. there exists an extension
\[
0 \to \mathcal{H} \to \mathcal{V} \to \mathbb{Z}(0) \to 0 \tag{2}
\]
in the category of admissible variations of mixed Hodge structure over \( P \) which induces the map \( \nu_{\zeta,L} \).

Given any point \( p \in \bar{P} \), the short exact sequence (2) induces a connecting homomorphism
\[
\delta : H^0(U \cap P, \mathbb{Z}(0)) \to H^1(U \cap P, \mathbb{H}_\mathbb{Z})
\]
for any analytic open neighborhood \( U \) of \( p \). Accordingly [BFNP], we define
\[
\sigma_p(\nu_{\zeta,L}) = [\delta(1)] \in (R^1_{\mathcal{J}_s} \mathcal{H})_p := \lim_{\to} H^1(U \cap P, \mathcal{H}_\mathbb{Z}) \tag{3}
\]
and let \( \text{sing}_p(\nu_{\zeta,L}) \) be the analogous invariant with values in \((R^1_{\mathcal{J}_s} \mathcal{H})_p\). The value of \( \text{sing}_p(\nu_{\zeta,L}) \) is independent of the choice of lift of \( \zeta \) to Deligne cohomology used in the construction of \( \nu_{\zeta,L} \).

**Definition 4.** The normal function \( \nu_{\zeta,L} \) is said to be singular at \( p \in \bar{P} \) if \( \text{sing}_p(\nu_{\zeta,L}) \neq 0 \) and singular on \( \bar{P} \) if the set of points \( p \in \bar{P} \) where \( \nu_{\zeta,L} \) is singular is non-empty.

**Conjecture 5.** Let \( L \) be a very ample line bundle on smooth projective variety \( X \) of even dimension. Let \( \zeta \) be a primitive, middle dimensional Hodge class on \( X \) which is not torsion. Then, there exists a positive integer \( d \) such that \( \nu_{\zeta,L^d} \) is singular on \( |L^d| \).

**Theorem 6.** [GG][BFNP][DM] Conjecture (5) holds for every triple \((X, L, \zeta)\) if and only if the Hodge conjecture holds for all smooth complex projective varieties.

Based upon the above, given a variation of Hodge structure of weight \(-1\) on a smooth algebraic variety \( S \) we define an admissible normal function to be a holomorphic section \( \nu \) of the associated bundle of intermediate Jacobians \( J(H) \) over \( S \) which arises from an extension of the form (2) in the category of admissible variations of mixed Hodge structure over \( S \). If \( S \subset \bar{S} \) then, mutatis mutandis, we can define \( \text{sing}_s(\nu) \) via (3).

By virtue of Theorem (6), proving the existence of singularities of normal functions becomes a central question in algebraic geometry. Ideally, one would like to find a system of equations on \( S \) which determine the singular locus of \( \nu \) on \( S \). Perhaps the simplest such equation is \( \nu = 0 \), which define a complex analytic subvariety
\[
Z = \{ s \in S \mid \nu(s) = 0 \}
\]
of \( S \).
Theorem 7. [BP3][Sc2][KNU3] Let $\mathcal{H}$ be a variation of Hodge structure of weight $-1$ over a smooth complex algebraic variety $S$. Then, the zero locus $Z$ of an admissible normal function $\nu: S \to J(\mathcal{H})$ is an algebraic subvariety of $S$.

The algebraicity of $Z$ and the existence of singularities of normal functions $\nu_\zeta$ considered by Griffiths and Green are linked by the following result of Christian Schnell:

Theorem 8. [Sc] Let $(X,L,\zeta)$ be a triple satisfying the hypothesis of Conjecture (5). Suppose that for $d$ sufficiently large, the zero locus $Z$ of the associated normal function $\nu_{\zeta,L^d}$ is positive dimensional. Then, $\nu_{\zeta,L^d}$ is singular on $|L^d|$.

The algebraicity of the zero locus also implies that it has a field of definition. As we shall now explain, in the context of admissible normal functions arising via families of cycles, the relationship between the field of definition of $Z$ and the field of definition of the cycles is closely related to the structure of the conjectured Bloch–Beilinson filtration on the Chow groups of smooth complex projective varieties.

More precisely, let $X$ be a smooth complex projective variety and 

$$\text{AJ}: CH^m_{\text{hom}}(X)_\mathbb{Q} \to J^m(X)_\mathbb{Q}$$

be the Abel–Jacobi map. Then, if the Bloch–Beilinson filtration exists 

$$F^2 CH^m(X)_\mathbb{Q} \subseteq \ker(\text{AJ})_\mathbb{Q}$$

In what follows, we shall say that an admissible normal function is motivated over a subfield $k$ of $\mathbb{C}$ if roughly speaking it arises from a family of cycles defined over $k$. For a precise definition, see section 4.5 of [KP].

Conjecture 9. (Conj. 81, [KP]) For every subfield $k$ of $\mathbb{C}$ which is finitely generated over $\bar{\mathbb{Q}}$, the zero locus of a $k$-motivated normal function $\nu: S \to J(\mathcal{H})$ is at most a countable union of subvarieties of $S$ defined over (possibly different) finite extensions of $k$.

In [L], James Lewis constructed a Leray filtration $L^* CH^m(X)_\mathbb{Q}$ which is known to coincide with the Bloch–Beilinson filtration under appropriate conditions.

Theorem 10. (Prop. 86, [KP]) Conjecture (9) holds for all $k$-motivated normal functions over curves if and only if $L^2 CH^m(X)_\mathbb{Q} = \ker(\text{AJ})_\mathbb{Q}$.

A normal function $\nu$ has a well defined infinitesimal invariant $\delta\nu$. In recent work [PS] with C. Schnell, we prove that the vanishing locus of $\delta\nu$ is a constructible subset of tangent bundle, and obtain the following result: If $\nu$ is a $k$-motivated normal function then the vanishing locus of $\delta\nu$ is defined over a finite extension of $k$. 
References


