ADMISSIBLE VARIATIONS OF MIXED HODGE STRUCTURE

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Abstract. We present versions of the orbit theorems for admissible variations of graded-polarized mixed Hodge structure and discuss their relationship with quantum cohomology in the case of a Calabi–Yau threefold.

During this talk, I shall describe extensions of the orbit theorems of E. Cattani, A. Kaplan and W. Schmid to admissible variations of graded-polarized mixed Hodge structures (VGPMHS) and their relationship with the quantum cohomology. In the case of a Calabi-Yau threefold, this relationship may be roughly summarized as follows:

1. The methods used in extending the orbit theorems to admissible VGPMHS yields a natural representation of the associate period map

\[ F(z_1, \ldots, z_n) : U^n \rightarrow M \]

of the form:

\[ F(z_1, \ldots, z_n) = e^{\sum_{j=1}^{n} z_j N_j} e^{\Gamma}.F_{\infty} \]

2. The “horizontal component” \( \Gamma_{-1} \) of \( \Gamma \) relative to the limiting mixed Hodge structure \( (F_{\infty}, W) \) gives rise to a Higgs field

\[ \partial X_{-1} = \sum_{j=1}^{n} N_j \otimes dz_j + \partial \Gamma_{-1} \]

on the product bundle \( V \times \Delta^{*n} \).

3. The connection

\[ d + \partial X_{-1} \]

defined by the A–model variation of a Calabi–Yau threefold on the product bundle \( V \times \Delta^{*n} \) coincides with the Dubrovin connection.
Part 1: Overview of Higgs Fields/Orbit Theorems.

**Higgs Fields:** Let \( E \to S \) be a holomorphic vector bundle. Then, an endomorphism valued 1-form \( \theta \) on \( E \) is said to be a Higgs field provided:

1. \( \theta \) is holomorphic.
2. \( \theta \wedge \theta = 0 \).

In the case of rank 2-vector bundles over a Riemann surface, such fields may be thought of as arising from the study of solutions to the self–dual Yang–Mills equations on \( \mathbb{R}^4 \) which are invariant under translation in two linearly independent directions.

**Orbit Theorems:** In order to state the Nilpotent Orbit Theorem, recall that the period map \( \phi \) of a variation of pure, polarized Hodge structure may be lifted to the upper half plane \( U \), resulting in a commutative diagram:

\[
\begin{array}{ccc}
U & \xrightarrow{F} & D \\
\exp(2\pi iz) \downarrow & & \downarrow \\
\Delta^* & \xrightarrow{\phi} & D/\Gamma
\end{array}
\]

Without loss of generality, assume the variation to have unipotent monodromy \( T = e^N \).

Then, as shown by W. Schmid, the map

\[ \psi(s) = e^{-zN}F(z) : \Delta^* \to \hat{D} \]

has the following properties:

- \( F_\infty := \lim_{s \to 0} \psi(s) \) exists as an element of \( \hat{D} \).
- The nilpotent orbit
  \[ F_{nilp}(z) = e^{zN}.F_\infty \]
  extends to a holomorphic, horizontal map \( \mathbb{C} \to \hat{D} \). Moreover, there exists \( \alpha > 0 \) such that \( F_{nilp}(z) \in D \) whenever \( \text{Im}(z) > \alpha \).
- There exists constants \( K > 0 \) and \( \beta \) such that for any choice of \( G_\mathbb{R} \)-invariant distance \( d \) on \( D \)
  \[ d(F(z), F_{nilp}(z)) \leq K \text{Im}(z)^\beta \exp(-2\pi \text{Im}(z)) \]
  whenever \( \text{Im}(z) > \alpha \).

In analogy with the pure case, the period map of a VGPMHS assumes values in a classifying space of GPMHS \( \mathcal{M} \) upon which a real Lie group \( G \) acts transitively. A direct
extension of Schmid’s proof of the Nilpotent Orbit Theorem to VGPMHS is however obstructed by the following two difficulties:

- The classifying spaces of GPMHS do not in general admit any $G$–invariant metrics.
- Some classifying spaces of GPMHS do not admit any metrics with negative holomorphic sectional curvature in horizontal directions.

As observed by A. Kaplan, the first of these obstructions may be overcome using a natural carried by the classifying spaces of GPMHS which is semi–homogeneous under the group action

$$G : M \rightarrow M$$

In overcoming the second difficulty, we use ideas of P. Deligne regarding gradings of mixed Hodge structures together with the work of Steenbrink–Zucker and Kashiwara on admissibility to obtain a proof par In analogy with the pure case, the period map of a VGPMHS assumes values in a classifying space of GPMHS $M$ upon which a real Lie group $G$ acts transitively. A direct extension of Schmid’s proof of the Nilpotent Orbit Theorem to VGPMHS is however obstructed by the following two difficulties:

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Part 2: Constructions/Preliminary Results

**Mixed Hodge Metric:** A bigrading of a mixed Hodge structure $(F, W)$ on $V$ consists of a direct sum decomposition

$$V = \bigoplus_{p,q} J^{p,q}$$

such that

$$F^p = \bigoplus_{a \geq p, b} J^{p,q}$$

$$W_k = \bigoplus_{a + k \leq k} J^{p,q}$$

for all indices $p$ and $k$. 
**Theorem (Deligne).** Every mixed Hodge structure \((F, W)\) defines a unique, functorial bigrading \(\{I^{p,q}\}\) of the underlying vector space such that

\[
I^{p,q} = I^{q,p} \mod \bigoplus_{a < q, b < p} I^{p,q}
\]

**Corollary.** Every mixed Hodge structure \((F, W)\) defines a unique, functorial grading \(Y\) of \(W\) by the rule:

\[
Y(v) = kv, \quad v \in \bigoplus_{a + b = k} I^{a,b}
\]

In particular, given a vector \(v \in V\) and a mixed Hodge structure \((F, W)\) on \(V\) we obtain a decomposition

\[
v = \sum_k v_k
\]

of \(v\) relative to the eigenvalues of \(Y = Y_{(F, W)}\).

Suppose now that \((F, W, S)\) is a graded-polarized mixed Hodge structure on \(V\), then we obtain a natural “mixed Hodge metric” on \(V\) by the formula:

\[
h_{(F, W)}(u, v) = \sum_k \langle Gr_k(u_k), Gr_k(v_k) \rangle_{FGW_k}
\]

where

- \(Gr_k : W_k \rightarrow Gr_k W\) denotes projection.
- \(\langle \cdot, \cdot \rangle_{FGW_k}\) denotes the pure Hodge metric defined by \((F, W, S)\) on \(Gr_k W\).

**Group Actions:** Let \(V\) be a complex vector space equipped with the following data

- A rational, increasing filtration \(W\).
- Rational, non-degenerate bilinear forms \(S_k\) on \(Gr_k W\) of parity \((-1)^k\).

Then, to each set of hodge numbers \(\{h^{p,q}\}\) we may associate a (possibly empty) classifying space

\[
\mathcal{M} = \mathcal{M}(W, S, h^{p,q})
\]

of graded-polarized mixed Hodge structures on \(V\) such that

- \((F, W, S)\) is a GPMHS.
- \(\dim_C F^p Gr_k W = \sum_{r \geq p} h^{r,k-r}\).

**Theorem.** The Lie group

\[
G = \{ g \in GL(V)^W \mid Gr(g) \in O(S, \mathbb{R}) \}
\]

acts transitively on \(\mathcal{M}(W, S, h^{p,q})\) by holomorphic diffeomorphisms.

At each point \(F \in \mathcal{M}\), the associate Deligne bigrading

\[
I^{p,q}_F = I^{p,q}_{(F, W)}
\]
may be used to construct a natural subspace
\[ q_F \subseteq \text{Lie}(G_C) \]
which is isomorphic to \( T_F(M) \) under the action
\[ u.\zeta = \frac{d}{dt} \zeta(e^{tu}.F) \bigg|_{t=0} \]
Consequently, the mixed Hodge metric induces a natural hermitian metric on the classifying spaces of GPMHS.

**Theorem.** The Lie group \( G_\mathbb{R} = G \cap GL(V_\mathbb{R}) \) acts by isometries on \( \mathcal{M} \).

**Proof.** The proof follows immediately from the following two observations:
- The Deligne bigrading has the following equivariance property:
  \[ g \in G_\mathbb{R} \implies I_{p,q}^g F = g I_{p,q}^F \]
- The group \( G_\mathbb{R} \) acts by isometries on \( Gr^W \).

**Remark.** The induced hermitian metric of \( \mathcal{M} \) has additional equivariance properties which may be used to compute the curvature of \( \mathcal{M} \).

**Relative Weight Filtration:** Each nilpotent endomorphism
\[ N : V \to V \]
on a finite dimensional vector space \( V \) determines a unique monodromy weight filtration
\[ 0 = W_0 \subseteq \cdots \subseteq W_{2\ell} = V \]
such that
- \( N : W_k \to W_{k-2} \) for each index \( k \).
- For each index \( k \), the map
  \[ N^k : Gr^W_\ell+k \to Gr^W_\ell-k \]
is an isomorphism.

The corresponding object for a filtered space \( (W,V) \) endowed with a nilpotent endomorphism \( N \) preserving \( W \) is the relative weight filtration
\[ ^r W = ^r W(N,W) \]
By definition, \( ^r W \) is the unique filtration with the following two properties
- \( N : ^r W_k \to ^r W_{k-2} \) for all \( k \).
- \( ^r W \) induces on each \( Gr^W_k \) the monodromy weight filtration of the map
  \[ Gr^W(N) : Gr^W_k \to Gr^W_k \]
However, the existence of the relative weight filtration is delicate and imposes many conditions on the pair \( (W,N) \).
Theorem (Deligne). Suppose the relative weight filtration \( \mathcal{W} = \mathcal{W}(N,W) \) exists and \( \mathcal{Y} \) is a grading of \( \mathcal{W} \) such that
\[
[\mathcal{Y},N] = -2N
\]
Then, there exists a unique grading \( \mathcal{Y} \) of \( \mathcal{W} \) such that
\begin{enumerate}
\item \( [\mathcal{Y},\mathcal{Y}] = 0 \).
\item \( N = N_0 + N_{-2} + \cdots \) when decomposed relative to the eigenvalues of \( \text{ad}\mathcal{Y} \).
\item \( (\text{ad}N_0)^k N_{-k} = 0 \) for all \( k > 0 \).
\end{enumerate}

Sketch. Start with any grading \( \mathcal{Y} \) of \( \mathcal{W} \) satisfying \( [\mathcal{Y},\mathcal{Y}] = 0 \) and observe the group
\[
G_0 = \{ g \in \text{GL}(V) \mid [g,\mathcal{Y}] = 0, \ (g-1) : W_k \to W_{k-1} \ \forall k \}
\]
acts transitively on the set of all such gradings. Deligne then shows via a method of “successive correction” that it is possible to select \( g \in G_0 \) such that \( \text{Ad}(g)\mathcal{Y} \) has the desired properties.

Admissibility: Let \( \mathcal{V} \to \Delta^* \) be a variation of pure, polarized Hodge structure. Then, as shown by the work of Cattani, Kaplan and Schmid:
\begin{itemize}
\item The limiting Hodge filtration \( F_\infty \) of \( \mathcal{V} \) exists.
\item Each element of the monodromy cone
\[
\mathcal{C} = \{ a_1N_1 + \ldots a_nN_n \mid a_1, \ldots, a_n > 0 \}
\]
determines the same monodromy weight filtration \( \mathcal{W} \) on \( \mathcal{V} \).
\item The pair \( (F_\infty,W) \) is a mixed Hodge structure relative to which each \( N_j \) is a \( (-1,-1) \) morphism.
\end{itemize}

For the remainder of this talk, we shall restrict our attention to a class of VGPMHS, called admissible by Steenbrink–Zucker and Kashiwara which share these properties (upon replacing the monodromy weight filtration by the relative weight filtration).

In particular, in the case of an admissible VGPMHS in one variable, the grading
\[
\mathcal{Y} = \mathcal{Y}(F_\infty,\mathcal{W})
\]
will satisfy \( [\mathcal{Y},N] = -2N \) since \( N \) is a \( (-1,-1) \) morphism of \( (F_\infty,\mathcal{W}) \).

Theorem. Let \( \mathcal{L} \to \Delta^* \) be a admissible VGPMHS. Then, the grading \( \mathcal{Y} \) associated to the pair \( (\mathcal{Y},N) \) by Deligne’s construction preserves \( F_\infty \).

Remark. Since \( \mathcal{Y} \) grades \( \mathcal{W} \), \( \mathcal{Y} \) grades \( \mathcal{W} \) and \( [\mathcal{Y},Y] = 0 \) it follows that \( \mathcal{Y}, \mathcal{Y} \) preserve both \( \mathcal{W} \) and \( \mathcal{W} \). In addition, \( \mathcal{Y} \) preserves \( F_\infty \) by construction.
**Asymptotic Structure:** Let \( \mathcal{L} \to \Delta^n \) be an admissible VGPMHS with associate map
\[
\psi : \Delta^n \to \mathcal{M}, \quad \psi(0) = F_{\infty}
\]
Then, because the action of the complexified Lie group
\[
G_{\mathbb{C}} : \mathcal{M} \to \mathcal{M}
\]
is transitive, each choice of a vector space decomposition
\[
\text{Lie}(G_{\mathbb{C}}) = \text{Lie}(G_{\mathbb{C}}^F) \oplus q
\]
determines a unique representation
\[
\psi(s_1, \ldots, s_n) = e^{\Gamma(s_1, \ldots, s_n)}, \quad \Gamma(0) = 0
\]
valid over a neighborhood of zero in \( \Delta^n \).

In order to select a natural complement \( q \) using the limiting mixed Hodge structure \((F_{\infty}, rW)\), define
\[
U^p_{\infty} = \bigoplus_q I^p,q(F_{\infty}, rW)
\]
and set
\[
\wp_a = \{ \alpha \in \text{Lie}(G_{\mathbb{C}}) \mid \alpha : U^p \to U^{p+a} \ \forall \ p \}
\]
We then have the following result:

**Lemma.** The graded, nilpotent subalgebra
\[
q_{\infty} = \bigoplus_{a < 0} \wp_a
\]
is a vector space complement to \( \text{Lie}(G_{\mathbb{C}}^F) \) in \( \text{Lie}(G_{\mathbb{C}}) \). Moreover, as \((-1, -1)\) morphism of \((F_{\infty}, rW)\), each monodromy logarithm \( N_j \) belongs to \( \wp_{-1} \).

Thus, as a consequence of this result, we obtain a natural representation of the period map
\[
F(z_1, \ldots, z_n) = e^{\sum_j z_j N_j} e^{\Gamma(s_1, \ldots, s_n)} F_{\infty}
\]
upon covering \( \Delta^n \) via
\[
s_j = e^{2\pi i z_j}
\]
It is then a consequence of the horizontality of \( F(z_1, \ldots, z_n) \) together with the graded structure of \( q_{\infty} \) that
\[
e^{-X} \partial e^X = \partial X_{-1}
\]
where
\[
X(z_1, \ldots, z_n) = \log \left( e^{\sum_{j=1}^n z_j N_j} e^{\Gamma(s_1, \ldots, s_n)} \right)
\]
and \( X_{-k} \) denotes the component of \( X \) taking values in \( \wp_{-k} \).
**Theorem.** The endomorphism valued 1-form $\partial X^{-1}$ defines a Higgs field on the product bundle $V \times \Delta^r \to \Delta^*$. 

**Proof.** Since $e^{-X} \partial e^X = \partial X^{-1}$ it follows that

$$\frac{\partial^2}{\partial z_i \partial z_j} e^X = e^X \frac{\partial X^{-1}}{\partial z_i} \frac{\partial X^{-1}}{\partial z_j} + e^X \frac{\partial^2 X^{-1}}{\partial z_i \partial z_j}$$

By equality of mixed partial derivatives, it therefore follows that

$$\partial X^{-1} \wedge \partial X^{-1} \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right) = \frac{\partial X^{-1}}{\partial z_i} \frac{\partial X^{-1}}{\partial z_j} - \frac{\partial X^{-1}}{\partial z_i} \frac{\partial X^{-1}}{\partial z_j} = 0$$

for all indices $i$ and $j$.

As discussed in greater length in [math.AG/9808106], the converse of this result is also true, namely:

**Theorem.** There is a bijective correspondence between holomorphic functions $\Gamma^{-1}: \Delta^n \to \varphi^{-1}, \quad \Gamma^{-1}(0) = 0$ satisfying the symmetry condition:

$$\partial X^{-1} \wedge \partial X^{-1} = 0, \quad \partial X^{-1} = \sum_{j=1}^n N_j \otimes dz_j + \partial \Gamma^{-1}$$

and admissible VGPMHS over $\Delta^*$ with given limiting data.

**Sketch.** The only difficult step in the proof is to show how to recover the function $\Gamma$ from the given data. To do this, one may use the differential equation

$$e^{-X} \partial e^X = \partial X^{-1}$$

and the graded structure of $q_\infty$ to recover $e^\Gamma$. Alternatively, it can be shown that

$$T_j(s_1, \ldots, s_n) = e^{-\Gamma} e^{-N_j} e^\Gamma$$

is the monodromy of the flat connection

$$\nabla = d + \partial X^{-1}$$

about the loop

$$(s_1, \ldots, e^{2\pi i t} s_j, \ldots, s_n), \quad 0 \leq t \leq 1$$
**Nilpotent Orbit Theorem:** Let $\mathcal{L} \to \Delta^*$ be an admissible VGPMHS (with unipotent monodromy) with associate period map

$$F(z) : U \to \mathcal{M}$$

Then, there exists a non-negative constant $\alpha$ such that:

1. The nilpotent orbit

$$F_{nilp}(z) = e^{zN}.F_{|\text{Im}fty}$$

assumes values in $\mathcal{M}$ when $y = \text{Im}(z) > \alpha$.

2. Relative to the $G_{\mathbb{R}}$–invariant distance on $\mathcal{M}$ determined by the mixed Hodge metric,

$$\text{Im}(z) > \alpha \implies d(F(z), F_{nilp}(z)) < Ky^\beta e^{-2\pi y}$$

where $K > 0$ and $\beta$ are constants depending only on the limiting data of $\mathcal{L}$.

**Sketch.** For simplicity, let us assume the limiting mixed Hodge structure $(F_{\infty}, ^rW)$ is split over $\mathbb{R}$ in the following sense:

$$\overline{I}^p_q(F_{|\text{Im}fty}, ^rW) = I^q_p(F_{\infty}, ^rW)$$

It then follows that

- The filtration

$$F_0 = e^{iN}.F_{\infty}$$

is an element of $\mathcal{M}$
- The gradings $^rY$ and $Y$ determined by the limiting data of $\mathcal{L}$ are defined over $\mathbb{R}$.

In particular, the nilpotent orbit $e^{zN}.F_{\infty}$ of $\mathcal{L}$ may be written

$$F_{nilp}(z) = y^{-\frac{1}{2}Y}e^{iN}y^\frac{1}{2}Y.F_{\infty}$$

$$= y^{-\frac{1}{2}Y}.F_0$$

since $^rY$ preserved $F_{\infty}$ and $[^rY, N] = -2N$.

Thus,

$$d(F(z), F_{nilp}(z)) = d(e^{iyN}e^{r(z)}.F_{|\text{Im}fty}, e^{iN}.F_{\infty})$$

$$= d(y^{-\frac{1}{2}Y}e^{iN}y^\frac{1}{2}Y e^{r(z)}.F_{\infty}, y^rY.F_0)$$

$$= d(y^{-\frac{1}{2}Y}e^{r(z)}.F_0, y^rY.F_0)$$

where

$$e^{\tilde{r}(z)} = \text{Ad}(e^{iN})\text{Ad}(y^rY)e^{r(z)}$$

As a consequence of the assumption that $(F_{\infty}, ^rW)$ is split over $\mathbb{R}$, it can be shown that

$$^rY - Y \in \text{Lie}(G_{\mathbb{R}})$$
and hence
\[
d(F(z), F_{nilp}(z)) = d(y^{-\frac{1}{2}e^\hat{F}(z)}.F_0, y^{-\frac{1}{2}e^\hat{F}}.F_0)
\]
since \([Y, Y] = 0\). Unraveling the definition of the mixed Hodge metric, it is now a straight forward computation to show the last equality implies the desired distance estimate.

**A–model:** Let \(X\) be a Calabi–Yau threefold with complexified Kähler moduli
\[
K_C(X) = \{ \omega \in H^2(X, \mathbb{C}) \mid Im(\omega) \text{ is a Kahler} \} / H^2(X, \mathbb{Z})
\]
and \(T_1, \ldots, T_n\) be a basis of \(H^2(X, \mathbb{Z})\) lying in the closure of the Kähler cone. Adopting the convention
\[
\omega(u_1, \ldots, u_n) = u_1T_1 + \ldots + u_nT_n, \quad Im(u_j) > 0
\]
and introducing the coordinates \(q_j = e^{2\pi i u_j}\) on \(\Delta^n\), the Gromov–Witten potential of \(X\) assumes the form
\[
\Phi(u_1, \ldots, u_n) = \left( \frac{1}{6} \int_X \omega^3 \right) + \Phi_{hol}(q_1, \ldots, q_n)
\]
where
\[
\Phi_{hol}(q_1, \ldots, q_n) = \frac{1}{(2\pi i)^3} \sum_{\beta \neq 0} \langle I_{0,0,\beta} \rangle e^{2\pi i \int_{\beta} \omega}
\]

Let \(V \to K_C(X)\) denote the trivial \(H^*(X, \mathbb{C})\) bundle over \(K_C(X)\) endowed with the Dubrovin connection
\[
\nabla = d + A, \quad A \frac{\partial}{\partial u_j} \alpha = T_j \ast \alpha
\]
encoding the small quantum product \(*\). As may be seen by direct computation, this connection is flat because the small quantum product is both commutative and associative. To construct a formal version of the B–model variation on \(V\), it therefore remains to describe

\begin{enumerate}
\item The Hodge filtration \(\mathcal{F}\).
\item The polarization \(Q\).
\item The integral structure \(\mathcal{V}_\mathbb{Z}\).
\end{enumerate}

over a neighborhood \(\Delta^n\) of a large radius limit point in \(K_C(X)\) with maximal unipotent monodromy.

In order to describe the integral structure of the A–model variation \(V \to \Delta^n\), let \(N_1, \ldots, N_n\) denote the monodromy logarithms of \(\nabla\) relative to the coordinates \((q_1, \ldots, q_n)\) and
\[
\nabla^\circ = \nabla - \frac{1}{2\pi i} \sum_{j=1}^n \frac{dq_j}{q_j} \otimes N_j
\]
denote the corresponding connection on the canonical extension \(V^\circ \to \Delta^n\). Then, a vector \(v \in \mathcal{V}_p\) is integral iff the parallel translate of \(v\) to \(\mathcal{V}_0\) belongs to \(H^*(X, \mathbb{Z})\).
To polarize $\mathcal{V}$, we pair $\alpha \in H^k(X, \mathbb{C})$ and $\beta \in H^{6-k}(X, \mathbb{C})$ by the rule

$$Q(\alpha, \beta) = (-1)^{k(k+1)/2} \int_X \alpha \wedge \beta$$

Finally, the Hodge filtration of $\mathcal{V}$ is given by

$$\mathcal{F}^p = \bigoplus_{a \leq 3-p} H^{a,a}(X, \mathbb{C})$$

**Theorem [Cox–Katz]**. The action of the monodromy logarithm $N_j$ on the central fiber of $\mathcal{V}^c$ is given by cup product with $T_j \in H^2(X, \mathbb{Z})$. Consequently, by the Hard Lefschetz Theorem, the monodromy weight filtration of $\mathcal{V} \to \Delta^* n$ is given by

$$W_k = \bigoplus_{2a \geq 6-k} H^{a,a}(X, \mathbb{C})$$

In addition, the following sections define a $\nabla^c$ flat frame of $\mathcal{V}^c$

$$\sigma_j = T_j - \sum \partial^2 \Phi_{hol} \frac{T^\vee}{\partial u_j} \frac{\partial \Phi}{\partial u_j} T^\vee_0, \quad \sigma_j = T^\vee_j$$

$$\sigma_0 = T_0 - \sum \partial T^\vee_0 \frac{T^\vee}{\partial u_j} \sigma_0, \quad \sigma_0 = T^\vee_0$$

where $T_0 = 1 \in H^0(X, \mathbb{C})$ and $^\vee : H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ denotes Poincaré duality.

**Theorem [Cox–Katz]**. The data $(\mathcal{V}_c, F, Q)$ defines a variation of pure, polarized Hodge structure of weight 3 over a neighborhood $\Delta^{*n}$ of a large radius limit point with maximal unipotent monodromy.

**Sketch.** Setting $F_\infty = F(0)$, the key step is to prove that $(F_\infty, W, Q)$ defines a polarized mixed Hodge structure.

Armed with these preliminaries, we can rewrite the Hodge filtration in terms of the $\nabla^c$-flat frame described above and thus obtain the “untwisted” period map $\psi : \Delta^n \to \tilde{D}$. Carrying out these computations, one finds that

$$\mathcal{F}^3 = \text{span}_\mathbb{C}(\sigma_0 + \sum \frac{\partial \Phi_{hol}}{\partial u_j} \sigma_0 - 2 \Phi_{hol} \sigma_j), \quad \mathcal{F}^1 = \mathcal{F}^2 \oplus \text{span}_\mathbb{C}(\sigma^j)$$

$$\mathcal{F}^2 = \mathcal{F}^3 \oplus \text{span}_\mathbb{C}(\sigma_j + \sum \frac{\partial^2 \Phi_{hol}}{\partial u_j} \sigma_0 - \frac{\partial \Phi_{hol}}{\partial u_j} \sigma_0), \quad \mathcal{F}^0 = \mathcal{F}^1 \oplus \text{span}_\mathbb{C}(\sigma^0)$$

where the index $j$ ranges from 1 to $n$.

Direct computation show the Deligne bigrading of the limiting mixed Hodge structure $(F_\infty, W)$ to be given by

$$I^{p,p} = H^{3-p,3-p}(X, \mathbb{C})$$

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and hence $\psi(q_1, \ldots, q_n) = e^{\sum q_i} F_\infty$ with

$$
\Gamma_{-1}(T_k) = \sum_{\ell=1}^n \frac{\partial^2 \Phi_{hol}}{\partial u_k \partial u_\ell} T_\ell^\vee \quad \Gamma_{-2}(T_k) = -\frac{\partial \Phi_{hol}}{\partial u_k} T_0^\vee \quad \Gamma_{-3}(T_k) = -2 \Phi_{hol} T_0^\vee
$$

The corresponding Higgs field $\partial X_{-1}$ of the A–model variation is therefore given by

$$
\partial X_{-1} = \sum_{j=1}^n \left( N_j + \frac{\partial \Gamma_{-1}}{\partial u_j} \right) \otimes du_j
$$

**Theorem.** For any vector $\alpha \in H^*(X, \mathbb{C})$,

$$
\partial X_{-1} \left( \frac{\partial}{\partial u_j} \right) \alpha = T_j \ast \alpha
$$

**Proof.** Since $N_j$ acts as wedge product by $T_j$ and $T_j \ast \alpha = T_j \wedge \alpha$ provided $\alpha$ has no component in $H^2(X, \mathbb{C})$, it will suffice to establish the claim for $\alpha \in H^2(X, \mathbb{C})$. However, by the previous formula:

$$
\partial X_{-1} \left( \frac{\partial}{\partial u_j} \right) T_k = T_j \wedge T_k + \sum_{\ell} \frac{\partial^3 \Phi_{hol}}{\partial u_j \partial u_k \partial u_\ell} T_\ell^\vee = T_j \ast T_k
$$

In particular, the symmetry condition $\partial X_{-1} \wedge \partial X_{-1} = 0$ implies $T_j \ast T_k = T_k \ast T_j$, since

$$
\partial X_{-1} \wedge \partial X_{-1} \left( \frac{\partial}{\partial u_j} , \frac{\partial}{\partial u_k} \right) T_0 = T_j \ast (T_k \ast T_0) - T_k \ast (T_j \ast T_0)
$$

$$
= T_j \ast T_k - T_k \ast T_j = 0
$$

Likewise, the symmetry condition $\partial X_{-1} \wedge \partial X_{-1} = 0$ implies the associativity of $\ast$ via the invariance of the invariance of the trilinear form

$$
\phi(\xi_a, \xi_b, \xi_c) = \int_X \partial X_{-1}(\xi_a) \circ \partial X_{-1}(\xi_b) \circ \partial X_{-1}(\xi_c) T_0
$$

under the natural action of the permutation group $S_3$ on the labels $a, b, c$.

Conversely, it can be shown that any solution of the WDVV (Witten–Dijgraaf–Verlinde–Verlinde) equation of the form

$$
\Phi = \left( \frac{1}{6} \int_X \omega^3 \right) + \Phi_{hol}, \quad \Phi_{hol}(0) = 0
$$

gives rise to a variation of Hodge structure via $(\ast)$. Thus, we have obtained the following result:

**Theorem.** There is a bijective correspondence between solutions to the WDVV equation of the form $(\ast)$ and variations of Hodge structure over $\Delta^{**n}$ governed by $(\ast)$. 

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