### 7.3. The Heat Equation

**Section Objective(s):**
- The Heat Equation (One-Space Dim).
- The IBVP: Dirichlet Conditions.
- The IBVP: Neumann Conditions.

**Remarks:**
- We solve a differential equation: the heat equation.
- This is a BVP and an IVP.
- We solve the heat equation using the separation of variables method.
- One first solves the BVP, which is an eigenfunction problem.
- The general solution of the BVP is a linear combination of all these eigenfunctions.
- One then uses the Fourier expansion formulas to find the unique combination of all eigenfunctions that satisfy the prescribed condition.
- We solve the heat equation for two types of boundary conditions: Dirichlet conditions and Neumann conditions.
7.3.1. The Heat Equation in (One-Space Dim).

**Definition 1.** The heat equation in one-dimension, for the function $u$ depending on $t$ and $x$ is

$$\frac{\partial u}{\partial t}(t, x) = k \frac{\partial^2 u}{\partial x^2}(t, x),$$

for $t \in [0, 1], x \in [0, L],\ $ where $k > 0$ is a constant.

**Remarks:**

- $u$ is the temperature of a solid material.
- $t$ is time, $x$ is space.
- $k > 0$ is the heat conductivity.
- The partial differential equation above has infinitely many solutions.
- We look for solutions satisfying both:
  - Boundary conditions.
  - Initial conditions.

Boundary Conditions: \quad Initial Conditions:
### 7.3.2. The IBVP: Dirichlet Conditions.

**Theorem 1 (Dirichlet).** The BVP for the one-space dimensional heat equation,

\[ \frac{\partial}{\partial t} u = k \frac{\partial^2}{\partial x^2} u, \]

BC:

\[ u(t, 0) = 0, \quad u(t, L) = 0, \]

where \( k > 0, \) \( L > 0 \) are constants, has infinitely many solutions.

Furthermore, for every continuous function \( f \) on \([0, L]\) satisfying

\[ f(0) = f(L) = 0, \]

there is a unique solution \( u \) of the boundary value problem above that also satisfies the \[ \] condition.

This solution \( u \) is given by the expression above, where the coefficients \[ \] are

### Remarks:

(a) This is an \[ \] Value Problem \[ \].

(b) The boundary conditions are called \[ \] boundary conditions.

**Remark:** The physical meaning of the initial-boundary conditions is simple.

1. The boundary conditions is to keep the \[ \] at the sides of the bar \[ \].

2. The initial condition is the \[ \] on the whole bar.

**Remark:** The proof is based on the \[ \] method.

1. Look for \[ \] solutions of the \[ \].

2. Linear combination of \[ \] solutions are solutions. (Superposition.)

3. Determine the free constants using the \[ \].
Proof of the Theorem:
Therefore, we got a simple solution of the heat equation BVP, 
\[ u_n(t, x) = c_n e^{k_n^2 t} \sin \left( \frac{n \pi x}{L} \right), \]
where \( n = 1, 2, \ldots \). Since the boundary conditions for \( u_n \) are homogeneous, then any linear combination of the \( u_n \) is also a solution of the heat equation with homogenous boundary conditions. So the most general solution of the BVP for the heat equation is 
\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{k_n^2 t} \sin \left( \frac{n \pi x}{L} \right). \]
Here the \( c_n \) are arbitrary constants. Now we look for the solution of the heat equation that in addition satisfies the initial condition 
\[ u(0, x) = f(x), \]
where \( f(0) = f(L) = 0 \). This initial condition is a condition on the constants \( c_n \), because 
\[ f(x) = \sum_{n=1}^{\infty} c_n \sin \left( \frac{n \pi x}{L} \right). \]
The problem now is, given \( f \), find the coefficients \( c_n \) such that the equation above holds.
One way to find the \( c_n \) is to use the Fourier formulas from the previous section. These formulas apply to functions on \([L, L]\). So, given \( f \) on \([0, L]\), we extend it to the domain \([L, L]\) as an odd function, 
\[ f_{\text{odd}}(x) = f(x) \text{ and } f_{\text{odd}}(x) = f(x), x \in [0, L]. \]
Since \( f(0) = 0 \), we get that \( f_{\text{odd}} \) is continuous on \([L, L]\). So \( f_{\text{odd}} \) has a Fourier series expansion. Since \( f_{\text{odd}} \) is odd, the Fourier series is a sine series 
\[ f_{\text{odd}}(x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n \pi x}{L} \right), \]
and the coefficients are given by the formula 
\[ b_n = \frac{2}{L} \int_{L}^{L} f_{\text{odd}}(x) \sin \left( \frac{n \pi x}{L} \right) \, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin \left( \frac{n \pi x}{L} \right) \, dx. \]
Since \( f_{\text{odd}}(x) = f(x) \) for \( x \in [0, L] \), then 
\[ c_n = b_n. \]
This establishes the Theorem. 
\[ \Box \]
Example 1: (Dirichlet): Find the solution to the initial-boundary value problem

\[ 4 \frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u, \quad t > 0, \quad x \in [0, 2], \]

with initial and boundary conditions given by

**IC:** \[ u(0, x) = \begin{cases} 
0 & x \in [0, \frac{2}{3}), \\
5 & x \in [\frac{2}{3}, \frac{4}{3}], \\
0 & x \in (\frac{4}{3}, 2], 
\end{cases} \]

**BC:** \[ \begin{cases} 
u(t, 0) = 0, \\
u(t, 2) = 0. \end{cases} \]

**Solution:**

We look for simple solutions of the form

\[ u(t, x) = v(t) w(x), \]

\[ 4 \dot{v}(t) = v(t) w''(x) + w'(x). \]

So, the equations for \( v \) and \( w \) are

\[ \dot{v}(t) = \frac{4}{v(t)} w'(x), \quad w''(x) + w'(x) = 0. \]

The solution for \( v \) depends on \( \mu \), and is given by

\[ v(t) = c e^{4t}, \quad c = v(0). \]

Next, we turn to the equation for \( w \), and we solve the BVP

\[ w''(x) + w'(x) = 0, \quad w(0) = w(2) = 0. \]

This is an eigenfunction problem for \( w \) and \( \mu \). This problem has solution only for \( \mu > 0 \), since only in that case the characteristic polynomial has complex roots. Let \( \mu = \mu^2 \), then

\[ p(r) = r^2 + \mu^2 = 0. \]

The general solution of the differential equation is

\[ w_n(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x). \]

The first boundary conditions on \( w \) implies

\[ 0 = w(0) = c_1, \quad w(x) = c_2 \sin(\mu x). \]

The second boundary condition on \( w \) implies

\[ 0 = w(2) = c_2 \sin(\mu 2), \quad c_2 \neq 0, \quad \sin(\mu 2) = 0. \]

Then, \( \mu^2 = n \pi^2, \) that is, \( \mu = n \pi \). Choosing \( c_2 = 1, \) we conclude,

\[ w_n(x) = \sin(n \pi x), \quad n = 1, 2, \ldots. \]

Using the values of \( n \) found above in the formula for \( v \) we get

\[ v_n(t) = c_n e^{1 \cdot \frac{n \pi^2}{4} t}, \quad c_n = v_n(0). \]

Therefore, we get

\[ u(t, x) = \sum_{n=1}^{\infty} c_n e^{1 \cdot \frac{n \pi^2}{4} t} \sin(n \pi x). \]
The initial condition is \( f(x) = u(0, x) = \begin{cases} 0 & x < 0 \\ 0 & x > 2 \end{cases} \). We extend this function to \([-2, 2]\) as an odd function, so we obtain the same sine function, \( f_{\text{odd}}(x) = f(x) \) and \( f_{\text{odd}}(x) = f(x) \), where \( x \in [0, 2] \).

The Fourier expansion of \( f_{\text{odd}} \) on \([-2, 2]\) is a sine series
\[
 f_{\text{odd}}(x) = \frac{1}{2} \sum_{n=1}^{\infty} b_n \sin(nc \pi x/2).
\]

The coefficients \( b_n \) are given by
\[
 b_n = \frac{2}{2/3 - 2/3} \int_{-2/3}^{2/3} f(x) \sin(nc \pi x/2) \, dx = \frac{10}{n \pi} \cos(nc \pi/3) \cos(nc \pi/3).
\]

So we get
\[
 b_n = \frac{10}{n \pi} \cos(nc \pi/3) \cos(nc \pi/3).
\]

Since \( f_{\text{odd}}(x) = f(x) \) for \( x \in [0, 2] \) we get that \( c_n = b_n \). So, the solution of the initial-boundary value problem for the heat equation contains
\[
 u(t, x) = \frac{10}{n \pi} \sum_{n=1}^{\infty} \cos(nc \pi/3) \cos(nc \pi/3) e^{(n \pi/4)^2 t} \sin(nc \pi x/2).
\]
7.3.3. The IBVP: Neumann Conditions.

**Theorem 2 (Neumann).** The BVP for the one-space dimensional heat equation,
\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},
\]
BC:
\[
\frac{\partial u}{\partial x}(t, 0) = 0, \quad \frac{\partial u}{\partial x}(t, L) = 0,
\]
where \( k > 0, L > 0 \) are constants, has \( \text{many solutions} \) \( \text{where} \) \( k > 0, L > 0 \) are constants, has \( \text{many solutions} \)

Furthermore, for every continuous function \( f \) on \([0, L]\) satisfying
\[
f(0) = f(L) = 0,
\]
there is a unique solution \( u \) of the boundary value problem above that also satisfies the \( \text{condition} \)

This solution \( u \) is given by the expression above, where the coefficients \( \text{are} \) \( \text{are} \)

**Remarks:**

(a) This is an \( \text{Initial-Boundary Value Problem (IBVP)} \).

(b) The boundary conditions are called \( \text{Neumann} \) boundary conditions.

**Remark:** The physical meaning of the initial-boundary conditions is simple.

(1) The boundary conditions is to keep the \( \text{heat flux} \) at the sides of the bar \( \text{constant} \).

(2) The initial condition is the \( \text{initial temperature} \) on the whole bar.

**Remark:** One can use \( \text{Dirichlet} \) conditions on one side and \( \text{Neumann} \) on the other side. This is called a \( \text{mixed} \) boundary condition.

**Remark:** The proof is based on the \( \text{separation of variables} \) method.
Proof of the Theorem:
The most general solution of the BVP for the heat equation is
\[ u(t, x) = c_0 + \sum_{n=1}^{\infty} c_n e^{k_n^2 t} \cos(n \pi x / L) \]
Here the \( c_n \) are arbitrary constants.

Now we look for the solution of the heat equation that in addition satisfies the initial condition
\[ u(0, x) = f(x), \]
where \( f(0) = f(L) = 0 \). This initial condition is a condition on the constants \( c_n \), because
\[ f(x) = u(0, x) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n \pi x / L). \]

The problem now is, given \( f \), find the coefficients \( c_n \) such that the equation above holds.

One way to find the \( c_n \) is to use the Fourier formulas from the previous section. These formulas apply to functions on \([0, L] \). So, given \( f \) on \([0, L] \), we extend it to the domain \([L, L]\) as an even function,
\[ f_{\text{even}}(x) = f(x) \text{ and } f_{\text{even}}(x) = f(x) \text{ for } x \in [0, L]. \]
We get that \( f_{\text{even}} \) is continuous on \([L, L]\). So \( f_{\text{even}} \) has a Fourier series expansion. Since \( f_{\text{even}} \) is even, the Fourier series is a cosine series
\[ f_{\text{even}}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n \pi x / L), \]
and the coefficients are given by the formula
\[ a_n = \frac{2}{L} \int_{0}^{L} f_{\text{even}}(x) \cos(n \pi x / L) \, dx = \frac{2}{L} \int_{0}^{L} f(x) \cos(n \pi x / L) \, dx, \quad n = 0, 1, 2, \ldots. \]

Since \( f_{\text{even}}(x) = f(x) \) for \( x \in [0, L] \), then \( c_n = a_n \) for \( n = 0, 1, 2, \ldots \). This establishes the Theorem.
Example 2: (Neumann): Find the solution to the initial-boundary value problem
\[ \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 3], \]
with initial and boundary conditions given by
\[
\text{IC: } u(0, x) = \begin{cases} 
7 & x \in \left[\frac{3}{2}, 3\right], \\
0 & x \in \left[0, \frac{3}{2}\right],
\end{cases} \quad \text{BC: } \begin{cases} 
\frac{d}{dt}u(t, 0) = 0, \\
\frac{d}{dt}u(t, 3) = 0.
\end{cases}
\]

Solution:
where we have added the trivial constant solution written as \( c_0/2 \). The initial condition is 
\[
 f(x) = u(0, x) = \begin{cases} 
 7x^2 \text{ for } -3 < x < 0, \\
 7x^2 \text{ for } 0 < x < 3. 
\end{cases}
\]

We extend \( f \) to \([-3, 3]\) as an even function 
\[
 f_{\text{even}}(x) = \begin{cases} 
 7x^2 \text{ for } -3 < x < 0, \\
 7x^2 \text{ for } 0 < x < 3. 
\end{cases}
\]

Since \( f_{\text{even}} \) is even, its Fourier expansion is a cosine series
\[
 f_{\text{even}}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n \pi}{3} x \right).
\]

The coefficient \( a_0 \) is given by
\[
 a_0 = \frac{2}{3} \int_{-3}^{3} f(x) \, dx = \frac{2}{3} \int_{-3}^{3/2} 7x^2 \, dx = \frac{2}{3} \left( 7 \cdot \frac{3}{2} \right) = 7.
\]

Now the coefficients \( a_n \) for \( n > 1 \) are given by
\[
 a_n = \frac{2}{3} \int_{-3}^{3} f(x) \cos \left( \frac{n \pi}{3} x \right) \, dx = \frac{2}{3} \int_{-3}^{3/2} 7x^2 \cos \left( \frac{n \pi}{3} x \right) \, dx.
\]

But for \( n = 2k \) we have that \( \sin \left( \frac{2k \pi}{2} \right) = \sin(k \pi) = 0 \), while for \( n = 2k+1 \) we have that 
\[
 \sin \left( \frac{2k+1 \pi}{2} \right) = \sin \left( \frac{\pi}{2} \right) = 1.
\]

Therefore 
\[
 a_{2k} = 0, \quad a_{2k+1} = \frac{2}{3} \cdot 7 \cdot \frac{\pi}{2} \sin \left( k \pi \right) = \frac{7}{2} \pi \sin \left( k \pi \right).
\]

We then obtain the Fourier series expansion of 
\[
 f_{\text{even}}(x) = \frac{7}{2} + \sum_{k=1}^{\infty} \frac{7}{2} \cos \left( \frac{k \pi}{3} x \right).
\]

But the function \( f \) has exactly the same Fourier expansion on \([0, 3]\), which means that 
\[
 c_0 = 7, \quad c_{2k} = 0, \quad c_{2k+1} = \frac{7}{2} \pi \sin \left( k \pi \right).
\]

So the solution of the initial-boundary value problem for the heat equation is
\[
 u(t, x) = \frac{7}{2} + \sum_{k=1}^{\infty} \frac{7}{2} \pi \sin \left( k \pi \right) e^{\left( \frac{k \pi}{3} \right)^2 t} \cos \left( \frac{k \pi}{3} x \right).
\]