7.3. The Heat Equation

Section Objective(s):

- The Heat Equation (One-Space Dim).
- The IBVP: Dirichlet Conditions.
- The IBVP: Neumann Conditions.

Remarks:

- We solve a ________________ differential equation: the heat equation.
- This is ______________ a ______________ and an ______________
- We solve the heat equation using the ____________________________________ method.
- One first solves the ________, which is an ________________________ problem.
- The _________________ solution of the _____________ is a linear combination of all these _____________________________
- One then uses the ____________________________ formulas to find the unique combination of all ____________________________ that satisfy the prescribed ________________________________
- We solve the heat equation for two types of boundary conditions: ______________ conditions and ________________ conditions.

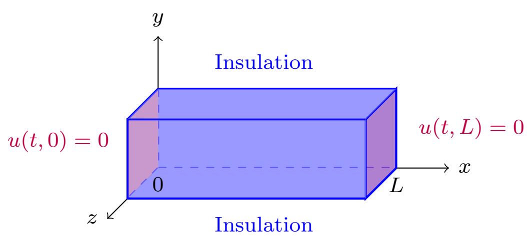
7.3.1. **The Heat Equation in (One-Space Dim).**

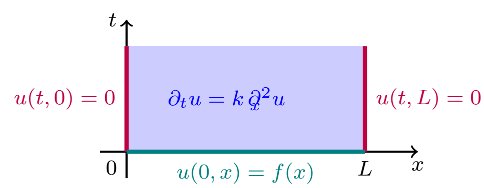
Definition 1. The ______________ equation in ________________ dimension, for the function u depending on __________________ is

where $k > 0$ is a constant.

Remarks:

- u is the __________________________ of a solid material.
- t is _____________, x is _______________.
- $k > 0$ is the ___________________________________.
- The partial differential equation above has _________________________ solutions.
- We look for solutions satisfying both:
  - _________________ conditions.
  - _________________ conditions.





Boundary Conditions: Initial Conditions:

7.3.2. The IBVP: Dirichlet Conditions.

Theorem 1 (Dirichlet). The BVP for the one-space dimensional heat equation,

where $k > 0$, $L > 0$ are constants, has ____________________ many solutions

Furthermore, for every continuous function f on $[0, L]$ satisfying

________________________________, there is a unique solution u of the boundary value problem above that also satisfies the _______________ condition

This solution u is given by the expression above, where the coefficients ____ are

Remarks:

(a) This is an ________________________________ Value Problem _______________.

(b) The boundary conditions are called ____________________ boundary conditions.

Remark: The physical meaning of the initial-boundary conditions is simple.

(1) The boundary conditions is to keep the _________________________ at the sides of the bar ____________________.

(2) The initial condition is the _________________________________ on the whole bar.

Remark: The proof is based on the ____________________________________ method.

(1) Look for ________________ solutions of the ______________.

(2) Linear combination of ________________ solutions are solutions. (Superposition.)

(3) Determine the free constants using the __________________________.

Proof of the Theorem:

□

EXAMPLE 1: (DIRICHLET): Find the solution to the initial-boundary value problem

$$4\,\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2],$$

with initial and boundary conditions given by

$$\text{IC:} \quad u(0,x) = \begin{cases} 0 & x \in \big[0, \dfrac{2}{3}\big), \\ 5 & x \in \big[\dfrac{2}{3}, \dfrac{4}{3}\big], \\ 0 & x \in \big(\dfrac{4}{3}, 2\big], \end{cases} \qquad \text{BC:} \quad \begin{cases} u(t,0) = 0, \\ u(t,2) = 0. \end{cases}$$

SOLUTION:

◁

7.3.3. **The IBVP: Neumann Conditions.**

Theorem 2 (Neumann). The BVP for the one-space dimensional heat equation,

where $k > 0$, $L > 0$ are constants, has ____________________ many solutions

Furthermore, for every continuous function f on $[0, L]$ satisfying
__________________________________, there is a unique solution u of the boundary value problem above that also satisfies the _______________ condition

This solution u is given by the expression above, where the coefficients ____ are

Remarks:

(a) This is an ________________________________ Value Problem _____________.

(b) The boundary conditions are called _____________________ boundary conditions.

Remark: The physical meaning of the initial-boundary conditions is simple.

(1) The boundary conditions is to keep the ____________________ at the sides of the bar ____________________.

(2) The initial condition is the _______________________________ on the whole bar.

Remark: One can use _______________ conditions on one side and ________________ on the other side. This is called a ________________ boundary condition.

Remark: The proof is based on the ____________________________________ method.

Proof of the Theorem:

□

Example 2: (Neumann): Find the solution to the initial-boundary value problem

$$\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 3],$$

with initial and boundary conditions given by

$$\text{IC:} \quad u(0, x) = \begin{cases} 7 & x \in \big[\dfrac{3}{2}, 3\big], \\ 0 & x \in \big[0, \dfrac{3}{2}\big), \end{cases} \qquad \text{BC:} \quad \begin{cases} u'(t, 0) = 0, \\ u'(t, 3) = 0. \end{cases}$$

Solution: