7.2. Overview of Fourier series

Section Objective(s):

- Vectors and the Dot Product in $\mathbb{R}^n$.
- Fourier Expansion of Functions.
- Odd or Even Functions.
- Sine and Cosine Series.

Remarks:

- We start with the Fourier expansion of a ____________________________.
- We review a few concepts:
  - The ______________________ of two vectors.
  - ___________________________ and _____________________________ vectors.
  - The decomposition of a vector in an _____________________________ basis.
- We then introduce the Fourier expansion of a ____________________________________.
- We need the following concepts:
  - The ______________________ of two functions.
  - ___________________________ and _____________________________ functions.
  - The decomposition of a function in an _____________________________ basis.
- We finish with two particular cases, the Fourier expansion of ____________________ functions and of ___________________ functions.

7.2.1. Vectors and the Dot Product in $\mathbb{R}^n$.

Remark: We review basic concepts about vectors in $\mathbb{R}^3$.

Definition 1. The ______________________ of $\boldsymbol{u} = \langle u_1, u_2, u_3\rangle$, $\boldsymbol{v} = \langle v_1, v_2, v_3\rangle$ is

Remark: The dot product above satisfies the following properties.

Theorem 1. For every $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in \mathbb{R}^3$ and every a, $b \in \mathbb{R}$ the following holds true:

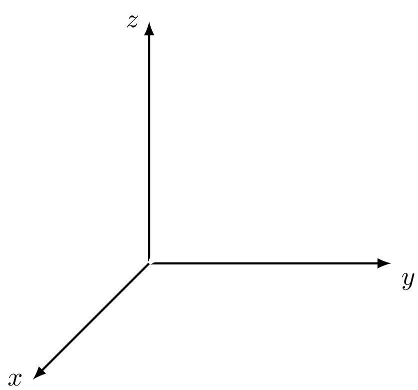
(a) ________________________ $\boldsymbol{u} \cdot \boldsymbol{u} = 0$ iff $\boldsymbol{u} = 0$; and $\boldsymbol{u} \cdot \boldsymbol{u} > 0$ for $\boldsymbol{u} \neq 0$.

(b) ________________________ $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$.

(c) ________________________ $(a\boldsymbol{u} + b\boldsymbol{v}) \cdot \boldsymbol{w} = a\,(\boldsymbol{u} \cdot \boldsymbol{w}) + b\,(\boldsymbol{v} \cdot \boldsymbol{w})$.

Theorem 2. The ______________________________ of two vectors $\boldsymbol{u}$, $\boldsymbol{v} \in \mathbb{R}^3$ is

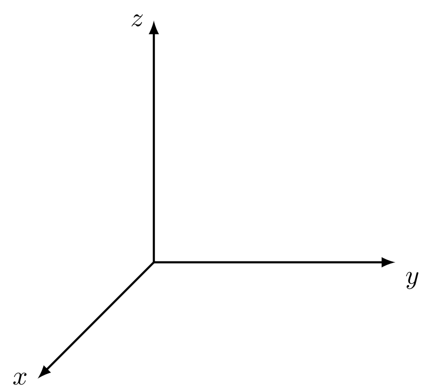
with $\|\boldsymbol{u}\|$, $\|\boldsymbol{v}\|$ the magnitude of the vectors, and $\theta \in [0, \pi]$ the angle in between them.



Remarks:
- The magnitude of a vector $\boldsymbol{u}$ can be written as
- A vector $\boldsymbol{u}$ is a unit vector iff

Theorem 3. The vectors $\boldsymbol{u}$, $\boldsymbol{v}$ are _____________________ iff _________________.

Example 1: The set $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is an ______________________________ of $\mathbb{R}^3$.



Orthonormal means:

- Orthogonality:

- Normality:

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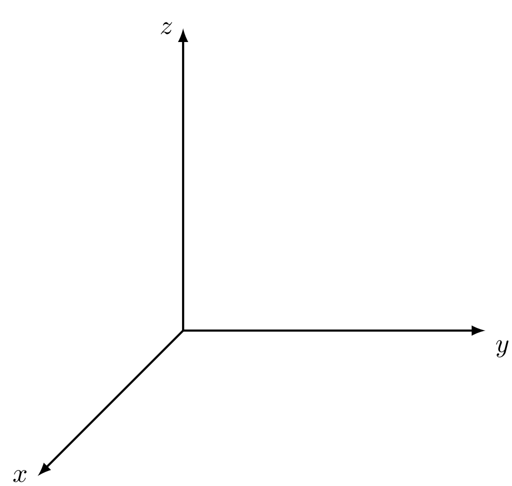
Theorem 4. (Fourier Expansion) The orthonormal set $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is an orthonormal ____________, that is, every $\boldsymbol{v} \in \mathbb{R}^3$ can be ____________________ as

The orthonormality of the vector set implies a formula for the vector components

Remark: The decomposition above allows us to introduce vector approximations.



Vector Approximations:

7.2.2. Fourier Expansion of Functions.

Remark: The ideas described above for vectors in $\mathbb{R}^3$ can be extended to functions.

Definition 2. The ____________________ of functions f, g on $[-L, L]$ is

Theorem 5. For every functions f, g, h and every a, $b \in \mathbb{R}$ holds,

(a) ____________________ $f \cdot f = 0$ iff $f = 0$; and $f \cdot f > 0$ for $f \neq 0$.

(b) ____________________ $f \cdot g = g \cdot f$.

(c) ____________________ $(a\,f + b\,g) \cdot h = a\,(f \cdot h) + b\,(g \cdot h)$.

Remarks:

- The ____________________ of a function f is

- A function f is a unit function iff ________________.

Definition 3. Two functions f, g are ________________ iff ____________.

Theorem 6. An example of an ________________________ in the space of continuous functions on $[-L, L]$ is

Remark: Often in the literature is used the following ____________________ set:

Remark: The orthogonality of the set above is a consequence of the following:

Theorem 7. (Orthogonality) The following relations hold for all $n,\, m \in \mathbb{N}$,

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\, \cos\Big(\frac{m\pi x}{L}\Big)\, dx = \begin{cases} \underline{\hspace{2cm}} & n \neq m, \\ \underline{\hspace{2cm}} & n = m \neq 0, \\ \underline{\hspace{2cm}} & n = m = 0, \end{cases}$$

$$\int_{-L}^{L} \sin\Big(\frac{n\pi x}{L}\Big)\, \sin\Big(\frac{m\pi x}{L}\Big)\, dx = \begin{cases} \underline{\hspace{2cm}} & n \neq m, \\ \underline{\hspace{2cm}} & n = m, \end{cases}$$

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\, \sin\Big(\frac{m\pi x}{L}\Big)\, dx = \underline{\hspace{2cm}}.$$

Proof:

Theorem 8. (Fourier Expansion) The orthogonal set

is an orthogonal _________ of the space of ____________________ functions on $[-L, L]$, that is, any continuous function on $[-L, L]$ can be ______________________ as

Moreover, the coefficients above are given by the formulas

Furthermore, if f is ______________________________________, then the function

satisfies ________________________ for all x where f is ____________________, while for all x_0 where f is _________________________ it holds

Example 2: Find the Fourier expansion of $f(x) = \begin{cases} \dfrac{x}{3}, & \text{for} \quad x \in [0,3] \\ 0, & \text{for} \quad x \in [-3,0). \end{cases}$

Solution:

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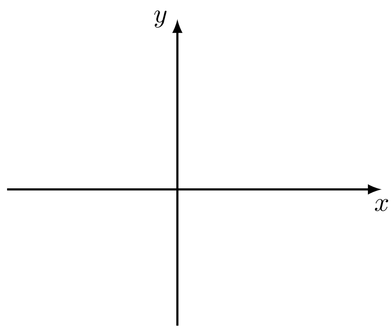
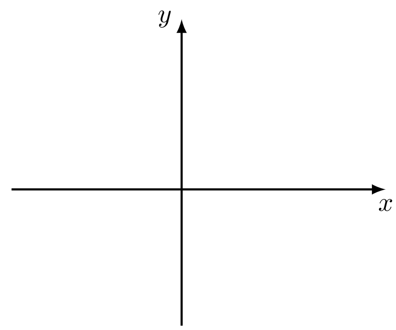
7.2.3. Odd or Even Functions.

Definition 4. A function f on $[-L, L]$ is:

- ____________ iff ______________________________ for all $x \in [-L, L]$;
- ____________ iff ______________________________ for all $x \in [-L, L]$.

EXAMPLE 3: The function $y = x^2$ is ________, while the function $y = x^3$ is _______.





Theorem 9. If f_e, g_e are even and h_o, ℓ_o are odd functions, then:

(1) $a\, f_e + b\, g_e$ is ___________ for all a, $b \in \mathbb{R}$.

(2) $a\, h_o + b\, \ell_o$ is ___________ for all a, $b \in \mathbb{R}$.

(3) $f_e\, g_e$ is ___________.

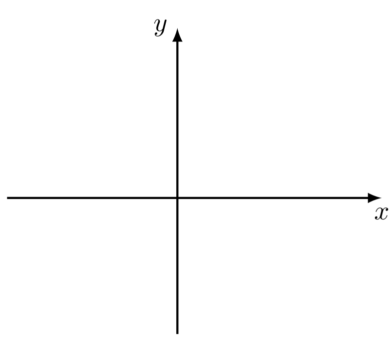
(4) $h_o\, \ell_o$ is ___________.

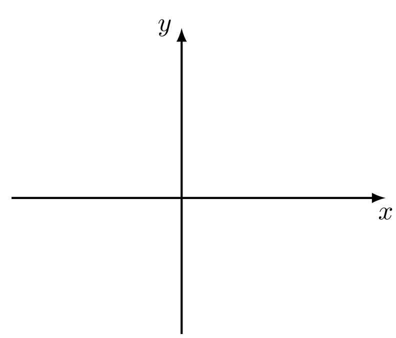
(5) $f_e\, h_o$ is ___________.

(6) $\displaystyle\int_{-L}^{L} f_e\, dx =$ ______________________________.

(7) $\displaystyle\int_{-L}^{L} h_o\, dx =$ ______________________________.

Remark:





7.2.4. **Sine and Cosine Series.**

Theorem 10. Let f be a function on $[-L, L]$ with a Fourier expansion

(a) If the function f is ______________, the Fourier series above is called a

___________________________, since _________________ and

(b) If the function f is _____________, then the Fourier series above is called a

_______________________, since _________________ and

Proof:

□

Example 4: Find the Fourier expansion of $f(x) = \begin{cases} 1, & \text{for} \quad x \in [0,3] \\ -1, & \text{for} \quad x \in [-3,0). \end{cases}$

Solution:

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