7.1. Simple Eigenfunction Problems

Section Objective(s):

- Two-Point Boundary Value Problems.
- Comparing IVP vs BVP.
- Eigenfunction Problems.

Remark:

- The main idea of this chapter is to solve ______________________________
- This is a ______________ differential equation.
- We need __________ main ideas to solve that equation.

  (1) ____ and

  ____

  (2) ____

- In this section we study the first idea: ______________________________

  ____.

7.1.1. Two-Point Boundary Value Problems.

Definition 1. A ***two-point boundary value problem*** (BVP) is the following: Find solutions to the differential equation

satisfying the boundary conditions (BC)

where b_1, b_2, $\tilde{b}_1$, $\tilde{b}_2$, y_1, y_2, x_1, x_2 are given and $x_1 \neq x_2$.

Remarks:

(a) The two boundary conditions are held at different points, ____________________.

(b) Both ______________________ may appear in the boundary condition.

EXAMPLE 1: We now show four examples of boundary value problems that differ only on the boundary conditions: Solve the different equation

$$y'' + a_1\, y' + a_0\, y = b(x)$$

with the boundary conditions at $x_1 = 0$ and $x_2 = 1$ given below.

(1a)

$$\text{Boundary Condition:} \quad \begin{Bmatrix} y(0) = y_1, \\ y(1) = y_2, \end{Bmatrix} \quad \text{which is the case} \quad \begin{Bmatrix} b_1 = \_\_, & b_2 = \_\_, \\ \tilde{b}_1 = \_\_, & \tilde{b}_2 = \_\_. \end{Bmatrix}$$

(1b)

$$\text{Boundary Condition:} \quad \begin{Bmatrix} y(0) = y_1, \\ y'(1) = y_2, \end{Bmatrix} \quad \text{which is the case} \quad \begin{Bmatrix} b_1 = \_\_, & b_2 = \_\_, \\ \tilde{b}_1 = \_\_, & \tilde{b}_2 = \_\_. \end{Bmatrix}$$

(1c)

$$\text{Boundary Condition:} \quad \begin{Bmatrix} y'(0) = y_1, \\ y(1) = y_2, \end{Bmatrix} \quad \text{which is the case} \quad \begin{Bmatrix} b_1 = \_\_, & b_2 = \_\_, \\ \tilde{b}_1 = \_\_, & \tilde{b}_2 = \_\_. \end{Bmatrix}$$

(1d)

$$\text{Boundary Condition:} \quad \begin{Bmatrix} y'(0) = y_1, \\ y'(1) = y_2, \end{Bmatrix} \quad \text{which is the case} \quad \begin{Bmatrix} b_1 = \_\_, & b_2 = \_\_, \\ \tilde{b}_1 = \_\_, & \tilde{b}_2 = \_\_. \end{Bmatrix}$$

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7.1.2. Comparing IVP vs BVP.

Definition 2. (IVP) Find a solution of $y'' + a_1\, y' + a_0\, y = 0$ satisfying the initial condition (IC)

Remarks:

- The variable t represents ____________________.
- The variable y represents ____________________.
- The IC are ____________________ and ____________________ at the initial time.

Definition 3. (BVP) Find a solution y of $y'' + a_1\, y' + a_0\, y = 0$ satisfying the boundary condition (BC)

Remarks:

- The variable x represents ______________________________.
- The variable y may represent ____________________________.
- The BC are ________________________ at two different ____________________.

Theorem 1. The equation $y'' + a_1\, y' + a_0\, y = 0$ with IC $y(t_0) = y_0$ and $y'(t_0) = y_1$ has a ____ for each choice of the IC.

Theorem 2. (BVP) The equation $y''+a_1\, y'+a_0\, y = 0$ with BC $y(0) = y_0$ and $y(L) = y_1$, with $L \neq 0$ and with $r_\pm$ roots of $p(r) = r^2 + a_1 r + a_0$ satisfy the following:

(A) If $r_+ \neq r_-$, reals, then the BVP above has a __________________________.

(B) If $r_\pm$ are complex, then the solution of the BVP above belongs to only one of the following three possibilities:

(i) There exists ____.

(ii) There exists ____.

(iii) There exists ____.

Proof of Theorem 2:

□

Example 2: Find all solutions to the BVPs $y'' + y = 0$ with the BCs:

$$\text{(a)} \quad \begin{cases} y(0) = 1, \\ y(\pi) = 0. \end{cases} \qquad \text{(b)} \quad \begin{cases} y(0) = 1, \\ y(\pi/2) = 1. \end{cases} \qquad \text{(c)} \quad \begin{cases} y(0) = 1, \\ y(\pi) = -1. \end{cases}$$

Solution:

$\triangleleft$

7.1.3. **Eigenfunction Problems.**

Remark: Let us recall the ***eigenvector*** problem of a square matrix: Given a square matrix A, find a number λ and a nonzero vector $\boldsymbol{v}$ solution of

Definition 4. An ***eigenfunction problem*** is the following: Given a linear operator $L(y) = a_2\, y'' + a_1\, y' + a_0\, y$, find a number λ and a nonzero function y solution of

and ____________________ boundary conditions at _______________,

Remarks:

- Notice that __________ is always _______________ of the BVP above.
- Eigenfunctions are the ________________________ of the BVP above.
- The eigenfunction problem is a BVP with ____________________ solutions.
- So, we look for ___ such that the operator ______________ has characteristic polynomial with ______________________.
- So, ___ is such that ________________ has ________________ solutions.
- We focus on the linear operator ____________________________.

EXAMPLE 3: Find all numbers λ and nonzero functions y solutions of the BVP

$$-y'' = \lambda\, y, \qquad \text{with} \qquad y(0) = 0, \quad y(L) = 0, \qquad L > 0.$$

SOLUTION:

◁

Example 4: Find the numbers λ and the nonzero functions y solutions of the BVP

$$-y'' = \lambda y, \qquad y(0) = 0, \qquad y'(L) = 0, \qquad L > 0.$$

Solution:

$\triangleleft$