6.3. Nonlinear Systems of Equations

Section Objective(s):

Part One:
- Two-Dimensional Nonlinear Systems.
- Critical Points and Linearization.
- The Hartman-Grobman Theorem.

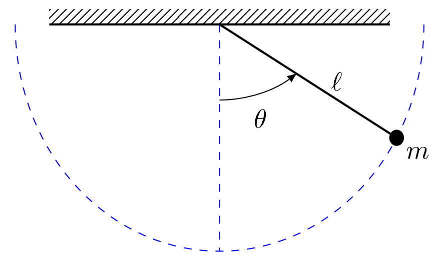
Part Two:
- Competing Species: Extinction.
- Competing Species: Coexistence.

Remarks:

- We know how to solve systems of ________________ differential equations.
- But systems of ______________________ differential equations are harder to solve.
- In this section we find ________________________ properties of the solutions to ______________________ systems.
- We first find the ______________________________ of the nonlinear system.
- We then find the behavior of solutions to ______________________ systems near the _______________________________ ___________________________________
- Finally, we _____________________________ the information from all the critical points to get a ________________________ phase portrait of solutions to the ______________________ system.
- We focus on two versions of the ________________________ species system:
  - The case when one species _______________________________
  - The case when both species _______________________________

6.3.1. **Two-Dimensional Nonlinear Systems.**

Example 1: (The Nonlinear Pendulum)



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Example 2: (Predator-Prey)
Let x be the predator and y be the prey. Then, the equation is

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Example 3: (Competing Species)
Let x_1 be the rabbit population and x_2 be the sheep population, both competing for the same food resources. The equation is

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6.3.2. Critical Points and Linearization.

Definition 1. A ______________ point of a system $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ is the end point of a vector $\boldsymbol{x}_c$ solution of

Remarks:

(a) Recall that $x = (x_1, x_2)$ is a ___________ on the x_1x_2-plane while $\boldsymbol{x} = \langle x_1, x_2 \rangle$ is a ____________ with origin at $(0,0)$ and end point at $x = (x_1, x_2)$.

(b) $\boldsymbol{x}_c$ is solution of $\boldsymbol{x}'(t) = \boldsymbol{f}(\boldsymbol{x})$, since

(c) In components, the field is $\boldsymbol{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, and the vector $\boldsymbol{x}_c = \begin{bmatrix} x_{c1} \\ x_{c2} \end{bmatrix}$ is solution of

When there are more than one critical point we write $\boldsymbol{x}_{c_i}$, with $i = 0, 1, 2, \cdots$.

Example 4: Find all the critical points of the two-dimensional (decoupled) system

$$\begin{aligned} x_1' &= -x_1 + (x_1)^3 \\ x_2' &= -2\,x_2. \end{aligned}$$

Solution:

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Definition 2. The ____________________________ of a 2×2 system $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ at a critical point given by $\boldsymbol{x}_c$ is the 2×2 linear system

where the ___________________________________ at $\boldsymbol{x}_c$ is,

Remark: : In components, the nonlinear system its linearization are

EXAMPLE 5: Find the linearization at every critical point of the nonlinear system

$$x_1' = -x_1 + (x_1)^3$$
$$x_2' = -2\, x_2.$$

SOLUTION:

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6.3.3. The Hartman-Grobman Theorem.

Remark: The linearization of a nonlinear system allow us to classify the critical points of nonlinear systems. linearization.

Definition 3. A critical point $\boldsymbol{x}^c$ of a 2×2 system $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ is:

(a) an _________ iff both eigenvalues of Df_c have negative real part;

(b) a _____________ iff both eigenvalues of Df_c have positive real part;

(c) a _____________ iff one eigenvalue of Df_c is positive and the other is negative;

(d) a _____________ iff both eigenvalues of Df_c are pure imaginary;

A critical point $\boldsymbol{x}^c$ is called ____________________ iff it belongs to cases (a-c), that is, the real part of all eigenvalues of Df_c are nonzero.

Theorem 1. (Hartman-Grobman) Consider a 2×2 nonlinear autonomous system,

with $\boldsymbol{f}$ continuously differentiable, and consider its linearization at a

_________________________ critical point given by $\boldsymbol{x}_c$,

Then, there is a neighborhood of $\boldsymbol{x}_c$ where all the solutions of the linear system

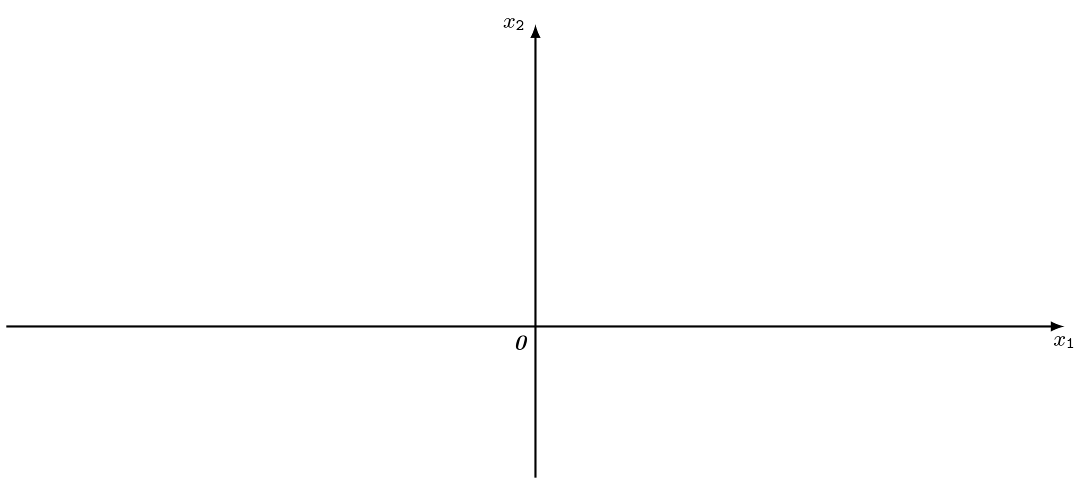
___________________________________ into solutions of the nonlinear system by a continuous, invertible, transformation.

Remark: The theorem above says that the phase portrait of the ______________________ at a ___________________ critical point is enough to determine the ___________________ ________________ of the phase portrait of the _________________ system near that critical point.

Example 6: Use the Hartman-Grobman theorem to sketch the phase portrait of

$$x_1' = -x_1 + (x_1)^3$$
$$x_2' = -2\,x_2.$$

Solution:



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6.3.4. **Competing Species: Extinction.**

Example 7: Find the linearization at every critical point of the competing species system

$$r' = r\,(3 - r - 2\,s),$$
$$s' = s\,(2 - s - r),$$

Remark: We call this model a rabbits-sheep model, where $r(t)$ is the rabbit population and $s(t)$ is the sheep population at the time t.

Solution:

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6.3.5. **Competing Species: Coexistence.**

Example 7: Find the linearization at every critical point of the competing species system

$$r' = r\,(1 - r - s),$$
$$s' = \frac{s}{4}\,(3 - 4\,s - 2\,r),$$

Remark: This is also a rabbits-sheep model, where $r(t)$ is the rabbit population and $s(t)$ is the sheep population at the time t.

Solution:

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