### 6.2. Two-Dimensional Phase Portraits and Stability

**Section Objective(s):**
- Real Distinct Eigenvalues.
  - $\lambda_- < \lambda_+ < 0$, Sink (Stable).
  - $0 < \lambda_- < \lambda_+$, Source (Unstable).
  - $\lambda_- < 0 < \lambda_+$, Saddle (Unstable).
- Complex Eigenvalues.

### 6.2.1. Review.

**Theorem 1.** The solutions of $\mathbf{x}' = A \mathbf{x}$, with $A$ a $2 \times 2$ matrix, depend on the eigenpairs of $A$, say $\lambda_{\pm}$, $\mathbf{v}^{(\pm)}$, as follows.

(a) If $\lambda_+ \neq \lambda_-$ and real, then $A$ is diagonalizable and

\[
\begin{align*}
    \mathbf{x}^+ (t) &= \mathbf{v}^+ e^{\lambda_+ t}, \\
    \mathbf{x}^- (t) &= \mathbf{v}^- e^{\lambda_- t},
\end{align*}
\]

(b) If $\lambda_+ = \alpha \pm \beta i$ and $\mathbf{v}^{(*)} = \mathbf{a} \pm \mathbf{b} i$, then $A$ is diagonalizable and

\[
\begin{align*}
    \mathbf{x}_1^+ (t) &= \mathbf{a} \cos(\beta t) + \mathbf{b} \sin(\beta t) e^{\alpha t}, \\
    \mathbf{x}_2^+ (t) &= \mathbf{a} \sin(\beta t) + \mathbf{b} \cos(\beta t) e^{\alpha t},
\end{align*}
\]

(c) If $\lambda_+ = \lambda_-$ and $A$ is diagonalizable, then $A = \lambda I$ and

\[
\begin{align*}
    \mathbf{x}^+ (t) &= \mathbf{v}^+ e^{\lambda t}, \\
    \mathbf{x}^- (t) &= (tv^+ + \mathbf{w}) e^{\lambda t},
\end{align*}
\]

where $\mathbf{v}^+ (A - \lambda I) = 0$ and $\mathbf{w} (A - \lambda I) = \mathbf{v}^+$.

(d) If $\lambda_+ = \lambda_-$ and $A$ is **not** diagonalizable, then

where
Example 1: Sketch a phase portrait and component plots of the fundamental solutions of $x' = Ax$, where the matrix $A$ is given by

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

Solution:

- Graph each component of $x_1(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ as function of $t$.

- Sketch a phase portrait.

Check the following Interactive Graph.
6.2.2. **Real Distinct Eigenvalues.**

**Case** \( \lambda < \lambda < 0: \textit{Sink (Stable)} \)

**Example 2:** Sketch a phase portrait of the solutions of the system,
\[
\mathbf{x}' = A \mathbf{x}, \quad A = \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix}.
\]

**Hint:** The eigenpairs of this matrix are \( \lambda_1 = -4, \mathbf{v}_1 = (1, 1) \), and \( \lambda_2 = -1, \mathbf{v}_2 = (-2, 1) \).

**Solution:**

Check the following **Interactive Graph.**
Remark: Use the Interactive Graph to find the phase portraits of the solutions to the following cases:

**Case** \(0 < \lambda < \lambda, \textbf{Source (Unstable)}\)

**Example 3:** Find the phase portrait of the solutions of the system

\[ \dot{x} = A x, \quad A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}, \]

**Hint:** The eigenpairs of this matrix are \(\lambda_1 = 4, \ v_1 = (1, 1), \) and \(\lambda_2 = 1, \ v_2 = (-2, 1).\)

**Case** \(\lambda < 0 < \lambda, \textbf{Saddle (Unstable)}\)

**Example 4:** Find the phase portrait of the solutions of the system

\[ \dot{x} = A x, \quad A = \begin{bmatrix} -2 & -3 \\ -3 & -2 \end{bmatrix}, \]

**Hint:** The eigenpairs of this matrix are \(\lambda_1 = -5, \ v_1 = (1, 1), \) and \(\lambda_2 = 1, \ v_2 = (-1, 1).\)
6.2.3. Complex Eigenvalues.

**Case** $\lambda_{\pm} = \alpha \pm \beta i$: **Spiral (Ellipse if $\alpha = 0$).**

- $\alpha > 0$, Source (Unstable).
- $\alpha = 0$, Center.
- $\alpha < 0$, Sink (Stable).

**Remark:** Use the Interactive Graph to help understand the phase portraits of the solutions to the following example.

**Example 5:** Find the phase portrait of the solutions of the system

$$x' = Ax, \quad A = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix}.$$ 

**Hint:** The eigenpairs of this matrix are $\lambda_{\pm} = -2 \pm 3i$, $v_{\pm} = (\pm i, 1)$. 
Remark: Summary:
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