6.2. Two-Dimensional Phase Portraits and Stability

Section Objective(s):

- Real Distinct Eigenvalues.
  - $\lambda_- < \lambda_+ < 0$, Sink (Stable).
  - $0 < \lambda_- < \lambda_+$, Source (Unstable).
  - $\lambda_- < 0 < \lambda_+ 0$, Saddle (Unstable).
- Complex Eigenvalues.

6.2.1. Review.

Theorem 1. The solutions of $\boldsymbol{x}' = A\,\boldsymbol{x}$, with A a 2×2 matrix, depend on the eigenpairs of A, say $\lambda_\pm$, $\boldsymbol{v}^{(\pm)}$, as follows.

(a) If $\lambda_+ \neq \lambda_-$ and real, then A is diagonalizable and

(b) If $\lambda_+ = \alpha \pm \beta i$ and $\boldsymbol{v}^{(+)} = \boldsymbol{a} \pm \boldsymbol{b}i$, then A is diagonalizable and

(c) If $\lambda_+ = \lambda_- = \lambda_0$ and A is diagonalizable, then $A = \lambda_0 I$ and

(d) If $\lambda_+ = \lambda_- = \lambda_0$ and A is ***not*** diagonalizable, then

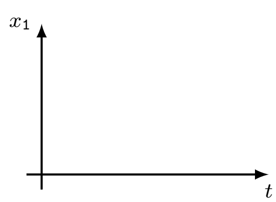
where

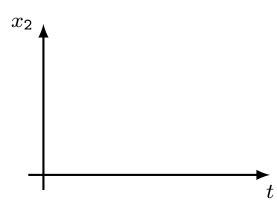
Example 1: Sketch a phase portrait and component plots of the of fundamental solutions of $\boldsymbol{x}' = A\,\boldsymbol{x}$, where the matrix A is given by

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

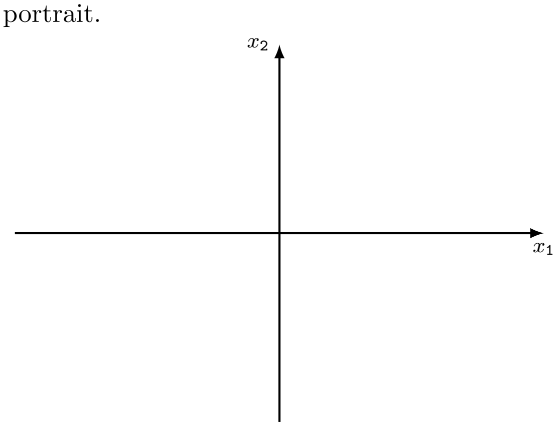
Solution:

- Graph each component of $\boldsymbol{x}_1(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ as function of t.





- Sketch a phase portrait.



Check the following **Interactive Graph**.

6.2.2. **Real Distinct Eigenvalues.**

Case $\lambda_- < \lambda_+ < 0$: ***Sink (Stable)***

Example 2: Sketch a phase portrait of the solutions of the system,

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix},$$

Hint: The eigenpairs of this matrix are $\lambda_1 = -4$, $\boldsymbol{v}_1 = \langle 1, 1 \rangle$, and $\lambda_2 = -1$, $\boldsymbol{v}_2 = \langle -2, 1 \rangle$.

Solution:

Check the following **Interactive Graph**.

Remark: Use the **Interactive Graph** to find the phase portraits of the solutions to the following cases:

Case $0 < \lambda_{\text{-}} < \lambda_{\text{+}}$ ***Source (Unstable)***

EXAMPLE 3: Find the phase portrait of the solutions of the system

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix},$$

Hint: The eigenpairs of this matrix are $\lambda_1 = 4$, $\boldsymbol{v}_1 = \langle 1, 1 \rangle$, and $\lambda_2 = 1$, $\boldsymbol{v}_2 = \langle -2, 1 \rangle$.

Case $\lambda_{\text{-}} < 0 < \lambda_{\text{+}}$ ***Saddle (Unstable)***

EXAMPLE 4: Find the phase portrait of the solutions of the system

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} -2 & -3 \\ -3 & -2 \end{bmatrix},$$

Hint: The eigenpairs of this matrix are $\lambda_1 = -5$, $\boldsymbol{v}_1 = \langle 1, 1 \rangle$, and $\lambda_2 = 1$, $\boldsymbol{v}_2 = \langle -1, 1 \rangle$.

6.2.3. Complex Eigenvalues.

Case $\lambda_{\pm} = \alpha \pm \beta i$: ***Spiral (Ellipse if*** $\alpha = 0$***).***

- $\alpha > 0$, **Source (Unstable).**
- $\alpha = 0$, **Center.**
- $\alpha < 0$, **Sink (Stable).**

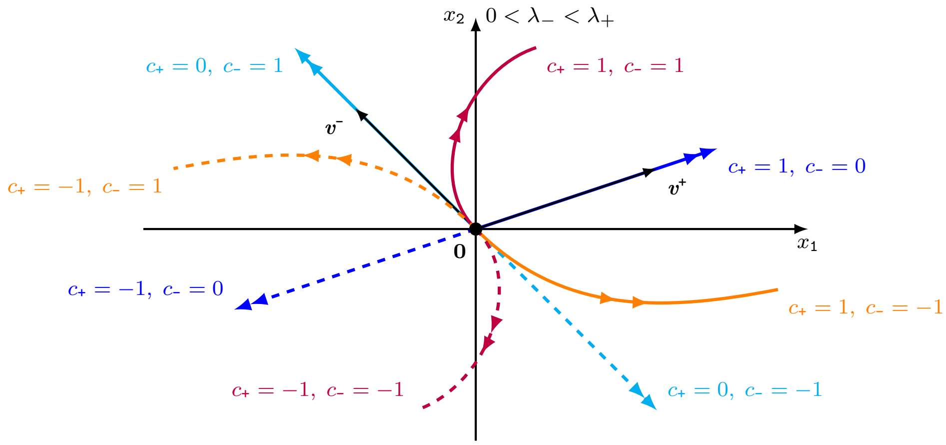
Remark: Use the **Interactive Graph** to help understand the phase portraits of the solutions to the following example.

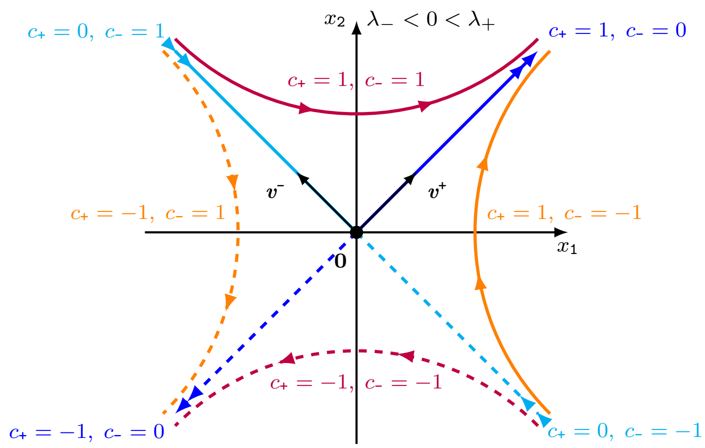
Example 5: Find the phase portrait of the solutions of the system

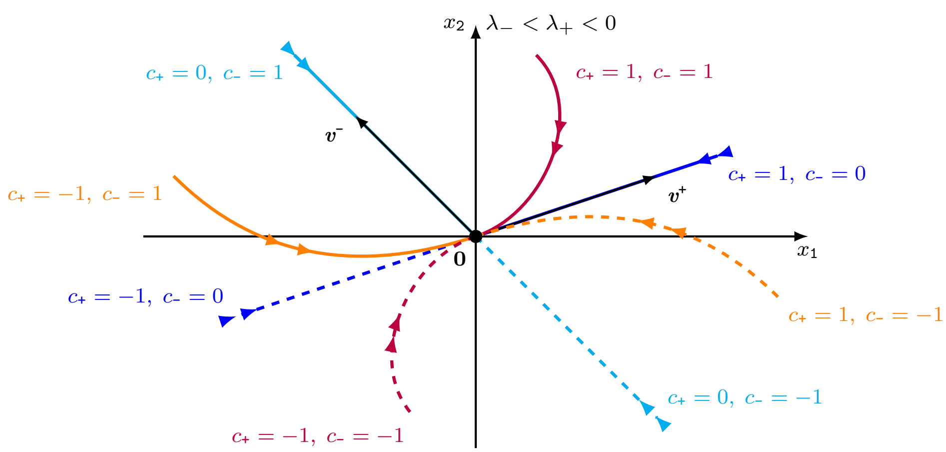
$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix}.$$

Hint: The eigenpairs of this matrix are $\lambda_{\pm} = -2 \pm 3i$, $\boldsymbol{v}_{\pm} = \langle \pm i, 1 \rangle$.

Remark: Summary:







Remark: Summary: