6.1. 2×2 Homogeneous Linear Differential Systems

Section Objective(s):

- 2×2 Linear Differential Systems.
- Diagonalizable Systems.
  - Real Distinct Eigenvalues.
  - Complex Eigenvalues.
  - Repeated Eigenvalues.
- Non-Diagonalizable Systems.
  - Repeated Eigenvalues.

Remarks:

- We introduce ________________ systems of linear ________________________________ equations.
- We focus on _______________________________ systems with ______________________ coefficients.
- If the homogeneous linear differential system is ________________________________, then we have a formula for ___________ the solutions.
- If the homogeneous linear differential system is __________________________________, then the formula above give only _____________ the solutions.
- The ________________________ of the solutions can be found generalizing ideas from ____ equations with ______________________ roots of their ____.

6.1.1. 2×2 Linear Differential Systems.

Definition 1. A 2×2 ***first order linear differential system*** is the equation

where the coefficient matrix A, the source vector $\boldsymbol{b}$, and the unknown vector $\boldsymbol{x}$ are

The system above is called:

- ______________________________ iff $\boldsymbol{b} = \boldsymbol{0}$,
- ____ iff A is constant,
- ____________________________________ iff A is diagonalizable.

Remarks:

- In this class we focus on _________________________ systems with

  ____

- Diagonal systems are very _______________ to solve.

EXAMPLE 1: Find functions x_1, x_2 solutions of the first order, 2×2, constant coefficients, homogeneous differential system

$$x_1' = 3x_1,$$
$$x_2' = 2x_2.$$

SOLUTION:

$\lhd$

6.1.2. **Diagonalizable Systems: Real Eigenvalues.**

Example 2: Now, we consider a system where the equations are **coupled**. Find functions x_1, x_2 solutions of the follwoing system of ODEs

$$
\begin{aligned}
x_1' &= x_1 + 3x_2, \\
x_2' &= 3x_1 + x_2.
\end{aligned}
$$

Solution:

$\triangleleft$

Theorem 1. (Homogeneous Diagonalizable Systems) If an $n \times n$ constant matrix A is

_____________________________________, with eigenpairs

then the general solution of $\boldsymbol{x}' = A\,\boldsymbol{x}$ is

Remark: Each function $\boldsymbol{x}_k(t) = e^{\lambda_k t}\,\boldsymbol{v}_k$ is solution of the system $\boldsymbol{x}' = A\,\boldsymbol{x}$, because

Example 3: Use the theorem above to find the general solution of the IVP

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} 3 & -2 \\ 10 & -6 \end{bmatrix}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Solution:

◁

6.1.3. Diagonalizable Systems: Complex Eigenvalues.

Remarks:

- A real matrix can have complex eigenvalues.
- But in this case, the eigenpairs come in conjugate pairs, $\lambda_- = \overline{\lambda}_+$, and $\boldsymbol{v}_- = \overline{\boldsymbol{v}}_+$.

Theorem 2. (Complex and Real Solutions) If a 2×2 matrix A has eigenpairs

where α, β, $\boldsymbol{a}$, and $\boldsymbol{b}$ real, then the equation $\boldsymbol{x}' = A\,\boldsymbol{x}$ has fundamental solutions

but it also has ***real-valued*** fundamental solutions

Proof of Theorem 2:

□

Example 4: Find real-valued fundamental solutions to the differential equation

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution:

◁

6.1.4. Diagonalizable Systems: Repeated Eigenvalues.

Remark: 2×2 linear differential systems with a ______________________ coefficient matrix with a ________________ eigenvalue are very simple to solve.

Theorem 3. (Diagonalizable with Repeated Eigenvalues) Every 2×2 diagonalizable matrix with a repeated eigenvalue λ_0 must have the form

Proof of Theorem 3:

□

Remark: : The differential equation $\boldsymbol{x}' = \lambda_0 I\, \boldsymbol{x}$ is already ________________.

$$\left.\begin{aligned} x_1' &= \lambda_0\, x_1 \\ x_2' &= \lambda_0\, x_2 \end{aligned}\right\} \quad \Rightarrow \quad \underline{\hspace{5cm}}$$

6.1.5. **Non-Diagonalizable Systems: Repeated Eigenvalues.**

EXAMPLE 5: Find fundamental solutions to the system

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$$

SOLUTION:

$\lhd$

Theorem 4. (Non-Diagonalizable with a Repeated Eigenvalue) If a 2×2 matrix A has a ____________________ eigenvalue λ_0 with ____________________ eigen direction determined by $\boldsymbol{v}_0$, then $\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t)$ has the linearly independent solutions

where the vector _________ is one solution of the algebraic linear system

Example 5-continued: Find the fundamental solutions of the differential equation

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

Solution:

$\triangleleft$