5.6. The Matrix Exponential

Section Objective(s):

- The Exponential of a Matrix.
  - Diagonal Matrices.
  - Diagonalizable Matrix.
- Properties of the Matrix Exponential.

Remarks:

- We know how to compute ______________________________ of matrices.
- We know how to compute ______________________________ of matrices
- With these operation it is possible to define ____________________ of matrices.
- We define __________________ of matrices using ______________________

5.6.1. **The Exponential of a Matrix.**

Review: Recall the definition of the exponential of real numbers.

- $f(x) = e^x$ is defined as:
  - For n natural number, ______________________________________.
  - Then, ________________, and for negative integers $-n$
  - Then, for rational numbers, _____________, with m, n integers,
  - Then, for ____________________ numbers x, is done by a limit,

  It is ___________ clear how to extend this definition to matrices.
- The exponential is the inverse of the natural log:

  and $\ln(y)$, is

  It is ___________ clear how to extend this definition to matrices.
- The exponential function can be defined also by its _________________________,

  This series expression ________________ generalized square matrices.

Definition 1. The ________________________ of a square matrix A is

Remark: It can be shown that the infinite sum above converges for all square matrices.

5.6.2. The Exponential of a Matrix: Diagonal Matrices.

Example 1: Compute e^A, where $A = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$.

Solution:

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Theorem 1. If $D = \operatorname{diag}\left[d_1, \cdots, d_n\right]$, then

5.6.3. The Exponential of a Matrix: Diagonalizable Matrices.

Remarks:

- The exponential of a ________________________ matrix is also simple to compute.
- We start computing powers of a ______________________________ matrix.

Theorem 2. If A is diagonalizable, with

$$A = PDP^{-1} = P\,\mathrm{diag}\big[a_{11}, \cdots, a_{nn}\big]\,P^{-1},$$

then

Proof of Theorem 2:

□

Theorem 3. The ____________________________ of a diagonalizable matrix A, with ____, is

Proof of Theorem 3:

□

EXAMPLE 2: Compute e^{At}, where $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ and $t \in \mathbb{R}$.

SOLUTION:

$\triangleleft$

5.6.4. Properties of the Matrix Exponential.

Remark: We now summarize the main properties of the matrix exponential.

Theorem 4. If A is an $n \times n$ matrix and s, t are real numbers, then

- Group property

- Inverse exponential

- Derivative of the exponential,

- If A, B are $n \times n$ matrices such that ________________________, then