5.3. Matrix Algebra

Section Objective(s):

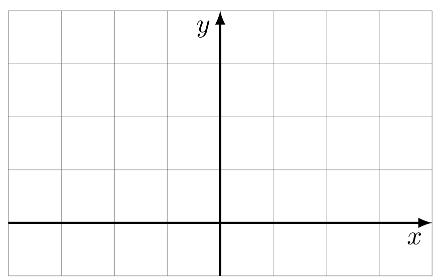
- The Determinant and Inverse of 2×2 Matrices.
- Equations for Matrices.
- Determinant of 3×3 Matrices.

Remarks:

- The row picture for algebraic linear systems

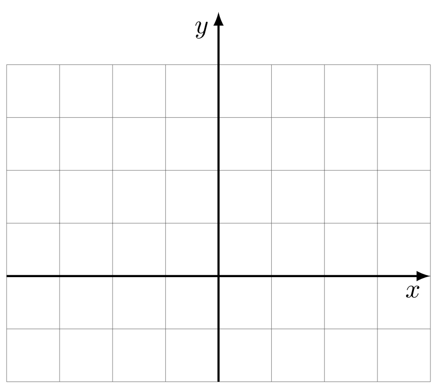
$$a\,x_1 + b\,x_2 = b_1,$$
$$c\,x_1 + d\,x_2 = b_2,$$

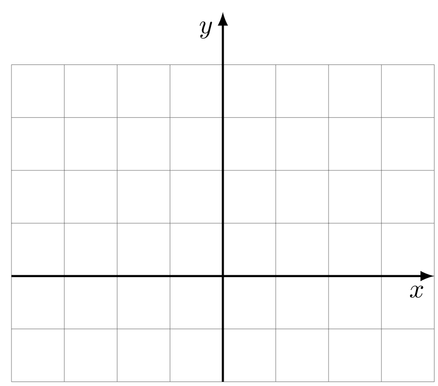
  is to intersect the solutions of each row in the system



- The column picture for algebraic linear systems above is

$$\begin{bmatrix} a \\ c \end{bmatrix} x_1 + \begin{bmatrix} b \\ d \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$





- The matrix picture of the linear system is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

5.3.1. The Determinant and Inverse of 2×2 Matrices.

EXAMPLE 1: Solve the linear system $A\boldsymbol{x} = \boldsymbol{b}$ given below and find the matrix $\tilde{A}$ such that the solution can be written as $\boldsymbol{x} = \tilde{A}\boldsymbol{b}$.

$$A\,\boldsymbol{x} = \boldsymbol{b}, \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

SOLUTION:

$\triangleleft$

Definition 1. The ______________ of a 2×2 matrix A

is defined when ______________________________________ and it is

If ____________________ the matrix has ________ inverse.

Remarks:

(a) The number $\det(A)$ is called the __________________________ of A.

(b) $\det(A)$ ________________________ whether A is invertible or not.

Example 2: Find the inverse of the given matrix $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ and verify that

$$(A^{-1})A = I_2, \qquad A(A^{-1}) = I_2, \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solution:

◁

Remark: $n \times n$ matrices can be invertible. The formula for the inverse of an $n \times n$ matrix is very complicated. However, we can still define what we mean by the inverse matrix.

Definition 2. An ________ matrix A is __________________ iff there is a matrix A^{-1} so that

5.3.2. **Equations for Matrices.**

Example 3: Compute the inverse of matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$.

Solution:

$\lhd$

Example 4: Find a matrix X such that $AXB = I$, where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solution:

$\lhd$

5.3.3. Determinant of 3×3 Matrices.

Definition 3. The ________________________ of a matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Remarks:

- A _______________ matrix is invertible iff ______________________________.
- The determinant of a _______________ matrix is defined recursively in terms of ________ determinants of _______________ matrices.
- The determinant of an _______________ matrix is defined recursively in terms of ________ determinants of ___________________________________ matrices.
- A _______________ matrix is invertible iff ______________________________.

Example 5: Compute the determinant of the 3×3 matrix, $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$.

Solution:

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