5.2. Vector Spaces

Section Objective(s):

- Vector Spaces, Subspaces, and Spans.
- Linear (In)dependence.
- Basis and Dimension.

Remarks:

- The ____________________ of linear algebraic equations originates the idea of a ______________________
- A ________________ is a smaller _____________________ inside a larger ______________________
- The __________ of a few vectors is the set of all ________________________ of these vectors.
- The __________ of a vectors creates ___________________.
- Vectors are ______________________________ if ____________________ is linear combination of the __________
- A ______ of V is the ________________________________ in V.
- A ______ of V is the __________________________________ in V.
- The __________________ of V is the number of ______________________.
- The __________________ of V measures ____________________ is V.

5.2.1. **Vector Spaces, Subspaces, and Spans.**

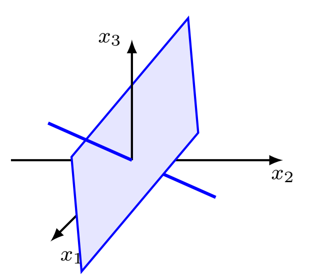
Definition 1. The ________________ $\mathbb{R}^n$, over $\mathbb{R}$, is the set of n-vectors with real components, together with the operation of ________________________,

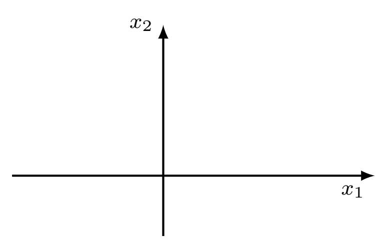
Remarks:

- The ________________ $\mathbb{C}^n$, over $\mathbb{C}$, is the set of n vectors with ____________ components, together with the ________________________ operation.
- We will use V to denote the ________________________________,
and $\mathbb{F}$ to denote the ________________________________

Definition 2. The subset $W \subseteq V$ of a vector space V over the field of scalars $\mathbb{F}$ is called a ____________________ iff for all $\boldsymbol{u}, \boldsymbol{v} \in W$ and all $a, b \in \mathbb{F}$,

EXAMPLE 1: Planes and lines ________________________ are subspaces of $\mathbb{R}^3$.





Example 2: Which of the following sets W are subspaces of the vector space V?

(1) $V = \mathbb{R}^2$, $W = \left\{ \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ such that } u_2 = 0 \right\}$.

(2) $V = \mathbb{R}^2$, $W = \left\{ \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ such that } u_2 = 1 \right\}$.

(3) $V = \mathbb{R}^2$, $W = \left\{ \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ such that } u_1 + u_2 = 0 \right\}$.

(4) $V = \mathbb{R}^3$, $W = \left\{ \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ such that } u_2 = 2u_3 \right\}$.

Remark: We finally introduce the definition of a span of a finite set of vectors.

Definition 3. The ______________ of a finite set ____________________________ in a vector space V over the field of scalars $\mathbb{F}$ is

Theorem 1. The Span(S) in a vector space V is a ____________________ of V.

Proof:

EXAMPLE 3: Give a geometric description of the following.

(1) Span($\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$) in $\mathbb{R}^3$, where $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$.

(2) Span($\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$) in $\mathbb{R}^3$, where $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$.

5.2.2. **Linear (In)dependence.**

Definition 4. A finite set of vectors ____________ in a vector space is called ____________________ iff there exists a set of scalars ____________, **not all of them zero**, such that,

The set ____________ is called ____________________ iff the **only solution** of the equation above is

Remarks:

- Linear dependence means ________________ is l.c. of the others.
- Linear independence means ________________ is l.c. of the others.

Example 4: Determine if the following sets are linearly independent and justify your claim.

(1) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$.

(2) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \right\}$.

5.2.3. Basis and Dimension.

Definition 5. A set $S \subset V$ is a ______________ of a vector space $V = \{\mathbb{R}^n, \mathbb{C}^n\}$ iff

(1) ____

and

(2) ____

EXAMPLE 5: Determine if the following sets provide bases for the given vector space.

(1) $V = \mathbb{R}^3$, $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(2) $V = \mathbb{R}^2$, $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$.

(3) $V = \mathbb{R}^3$, $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \right\}$.

Theorem 2. The ________________________________ in any basis of a vector space $V = \{\mathbb{R}^n, \mathbb{C}^n\}$ is ______________________________ in any other basis of V.

Definition 7. The ____________________ of a vector space $V = \{\mathbb{R}^n, \mathbb{C}^n\}$ is n,

____

Example 6: Give an example of two different bases of $\mathbb{R}^2$.

Example 7: Determine the dimension of the vector space given by

$$W = \text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} \right\}$$