3.5. Convolutions and Solutions

Section Objective(s):
- The Convolution of Two Functions.
- Main Properties of the Convolution.
- The Solution Decomposition Theorem.

Remarks:

- We introduce a new operation between two function, the ____________________.
- The ____________________________ is a __________________________________ of two functions.
- We know that ____
- ____________________________ is such that
- ____________________________ is defined for __________________________________.
- ___________________________________ is the ____.

3.5.1. The Convolution of Two Functions.

Definition 1. The ***convolution*** of functions f and g is a function $f * g$ given by

Remark: The convolution is defined even when either f and g is a ______________________.

EXAMPLE 1: Find $f * g$ the convolution of the functions $f(t) = b(t)$ and $g(t) = b(t)$, where we denoted $b(t) = u(t) - u(t-1)$, the bump function on $[0, 1]$.

Interactive Graph: Convolution of Bumps

SOLUTION:

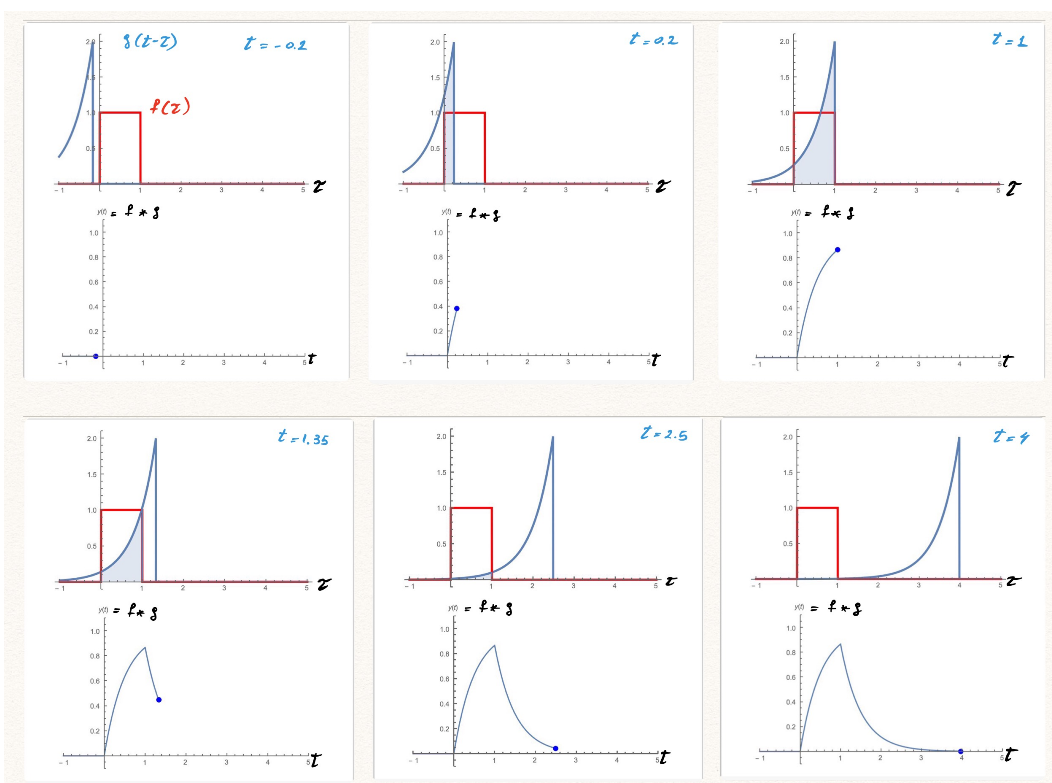
◁

Example 2: Graph the convolution of

$$f(\tau) = u(\tau) - u(\tau - 1),$$
$$g(\tau) = \begin{cases} 2\,e^{-2\tau} & \text{for} \quad \tau \geqslant 0 \\ 0 & \text{for} \quad \tau < 0. \end{cases}$$

Interactive Graph: Convolution of Bump and Exponential (Slow)

Solution:



◁

3.5.2. Main Properties of the Convolution.

Theorem 1. (Laplace Transform) If $\mathcal{L}[f]$ and $\mathcal{L}[g]$, exist, then

Remark: This is the origin of the convolution operation. Since

people were interested in finding a function h such that

The answer is,

Idea of the Proof:

Other Properties of Convolutions:

Theorem 2. For every piecewise continuous functions f, g, and h, hold:

(i) Commutativity:

(ii) Associativity:

(iii) Distributivity:

(iv) Neutral element:

(v) Identity element:

Example 3: Find the function g such that $f(t) = \int_0^t \sin(4\tau)\, g(t-\tau)\, d\tau$ has the Laplace transform $\mathcal{L}[f] = \dfrac{s}{(s^2+16)((s-1)^2+9)}$.

Solution:

$\lhd$

3.5.3. The Solution Decomposition Theorem.

Theorem 3. (Solution Decomposition) The solution of

$$L(y) = g(t), \quad y(0) = y_0, \quad y'(0) = y_1,$$

where $L(y) = y'' + a_1\, y' + a_0\, y$ has constant coefficients, can be decomposed as

where y_h is the solution of the homogeneous initial value problem

and y_δ is the ____ of L.

Remarks:

(1) The solution decomposition above can be written in the equivalent way

(2) Recall that the impulse response function is the solution of

(3) Recall that the impulse response function can be written as

Example 4: Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = g(t), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution:

◁