3.4. Generalized Sources

Section Objective(s):

- The Dirac's Delta.
- Applications and Properties.
- The Impulse Response Function.

Remarks:

- The Dirac's delta is the main example of what it is called a

  ____.

- Introduced by ________________________ while studying

  ____.

- Dirac's Delta satisfies:
  - It is _________________________________ except __________________.
  - At _____________ the Dirac's delta ____________________.
  - The ___________________ of Dirac's delta is ____________________________.

  __________________________ has these properties.

- Dirac's delta is ____ of functions.

3.4.1. **The Dirac Delta.**

Definition 1. The ***Dirac delta*** generalized function is the limit

for every fixed $t \in \mathbb{R}$ of the sequence functions $\{\delta_n\}_{n=1}^{\infty}$,

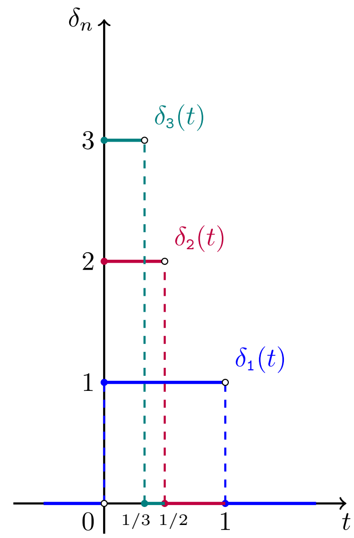
Remark: The sequence of bump functions introduced above can be rewritten as follows,

$$\delta_n(t) = \begin{cases} \underline{\hspace{2cm}}, & t < 0 \\ \underline{\hspace{2cm}}, & 0 \leqslant t < \dfrac{1}{n} \\ \underline{\hspace{2cm}}, & t \geqslant \dfrac{1}{n}. \end{cases}$$

We then obtain the equivalent expression,

$$\delta(t) = \begin{cases} \underline{\hspace{2cm}} & \text{for} \quad t \neq 0, \\ \underline{\hspace{2.5cm}} & \text{for} \quad t = 0. \end{cases}$$

Remark: There are infinitely many sequences $\{\delta_n\}$ of functions with the Dirac delta as their limit as $n \to \infty$.



Interactive Graph: Dirac's Delta.

Remarks:

(a) The Dirac delta is $\underline{\hspace{8cm}}$ on the domain $\underline{\hspace{4cm}}$.

(b) The Dirac delta is $\underline{\hspace{7cm}}$ on $\underline{\hspace{2.5cm}}$.

Theorem 1. Every function in the sequence $\{\delta_n\}$ above satisfies

3.4.2. **Applications and Properties.**

Applications:

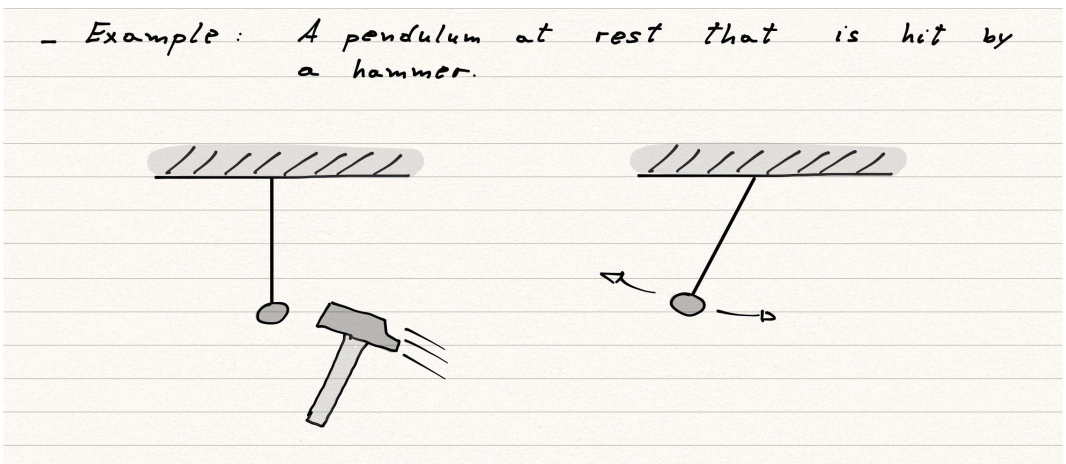
(a) Dirac's delta generalized function is useful to describe

____.

(b) An impulsive force transfers a ____________________________________

in an ____.

(c) For example, a pendulum at rest that is hit by a hammer.



Main Properties:

Theorem 2. If f is continuous on (a, b) and $c \in (a, b)$, then

Proof of Theorem 2:

□

Theorem 3. For all $s \in \mathbb{R}$ holds

$$\mathcal{L}[\delta(t-c)] = \begin{cases} \underline{\hspace{4cm}} & \text{for} \quad c \geqslant 0, \\ \underline{\hspace{4cm}} & \text{for} \quad c < 0. \end{cases}$$

Proof of Theorem 4.4.5:

□

3.4.3. The Impulse Response Function.

Definition 2. The ***impulse response function*** at the point $c \geqslant 0$ of the linear operator

with a_1, a_0 constants, is the solution y_δ of

Theorem 4. The function y_δ is the impulse response function at $c \geqslant 0$ of the constant coefficients operator $L(y) = y'' + a_1\, y' + a_0\, y$ iff holds

where ____ of L.

Proof of Theorem 4:

□

EXAMPLE 1: Find the solution y to the initial value problem

$$y'' - y = \delta(t-3), \qquad y(0) = 0, \qquad y'(0) = 0.$$

SOLUTION:

$\triangleleft$