3.1. Introduction to the Laplace Transform

Section Objective(s):

- The Laplace Transform.
- Main Properties.
- Solving a Differential Equation.

Remarks:

- The Laplace Transform (LT) method introduces a ____________________________ to solve differential equations.
- The idea is to use ____.
- Because of that the LT changes ___________________ into ___________________________.
- So, LT changes ___________________ equations into ________________ equations.

- With the LT we can solve differential equations with ___________________________.
- Examples include:
  - Solve for the motion of objects hit by ____________________________.
  - Solve for the current in an electric circuit having switches ___________________________.
- The Undetermined Coefficients Method is _____________________________________ to solve differential equations with such general sources.

3.1.1. The Laplace Transform.

Definition 1. The ***Laplace transform*** of a function f on $D_f = [0, \infty)$ is

defined for all $s \in D_F \subset \mathbb{R}$ where the ______________________.

Remarks:

(a) Transformation notations for the Laplace transform: ______________.

(b) Recall the definition of improper integrals:

$$\int_0^\infty g(t)\,dt = \qquad\qquad .$$

3.1.2. Main Properties.

Linearity

Theorem 1. (Linearity) If $\mathcal{L}[f]$ and $\mathcal{L}[g]$ exist, then for all $a,\, b \in \mathbb{R}$ holds

Proof of Theorem 1:

$\square$

Remark: $\mathcal{L}[c\, f(t)] = c\,\mathcal{L}[f(t)$, but ______________________.

Derivatives into Multiplication

Theorem 2. (Derivative into Multiplication) If both f and f' are continuous and $|f(t)| \leqslant k\, e^{at}$, with $k, a > 0$, all conditions on $[0, \infty)$, then $\mathcal{L}[f']$ exists for $s > a$ and

Proof of Theorem 2:

□

Exercise: Use the formula above to compute the LT of second (and higher) derivatives,

$$\mathcal{L}[f''] = \qquad\qquad\qquad .$$

EXAMPLE 1. (COMPUTING A LT): Compute $\mathcal{L}[e^{at}]$, where $a \in \mathbb{R}$.

SOLUTION:

$\triangleleft$

Laplace Transform Table: We collect the LT of simple functions.

$f(t)$	$F(s) = \mathcal{L}[f(t)]$	D_F
$f(t) = 1$	$F(s) = \dfrac{1}{s}$	$s > 0$
$f(t) = e^{at}$	$F(s) = \dfrac{1}{(s-a)}$	$s > a$
$f(t) = t^n$	$F(s) = \dfrac{n!}{s^{(n+1)}}$	$s > 0$
$f(t) = \sin(at)$	$F(s) = \dfrac{a}{s^2 + a^2}$	$s > 0$
$f(t) = \cos(at)$	$F(s) = \dfrac{s}{s^2 + a^2}$	$s > 0$
$f(t) = \sinh(at)$	$F(s) = \dfrac{a}{s^2 - a^2}$	$s > \|a\|$
$f(t) = \cosh(at)$	$F(s) = \dfrac{s}{s^2 - a^2}$	$s > \|a\|$
$f(t) = t^n e^{at}$	$F(s) = \dfrac{n!}{(s-a)^{(n+1)}}$	$s > a$
$f(t) = e^{at}\sin(bt)$	$F(s) = \dfrac{b}{(s-a)^2 + b^2}$	$s > a$
$f(t) = e^{at}\cos(bt)$	$F(s) = \dfrac{(s-a)}{(s-a)^2 + b^2}$	$s > a$
$f(t) = e^{at}\sinh(bt)$	$F(s) = \dfrac{b}{(s-a)^2 - b^2}$	$s - a > \|b\|$
$f(t) = e^{at}\cosh(bt)$	$F(s) = \dfrac{(s-a)}{(s-a)^2 - b^2}$	$s - a > \|b\|$

3.1.3. **Solving a Differential Equation.**

EXAMPLE 2. (SOLVING AN IVP): Use the Laplace transform to find y, solution of

$$y' = -5\,y, \qquad y(0) = 2.$$

SOLUTION:

◁