2.3. Nonhomogeneous Equations

Section Objective(s):
Part 1:
- The General Solution Theorem(NH).
- The Undetermined Coefficients Method.

Part 2:
- The Variation or Parameters Method.

Remarks:

- If y_1 and y_2 are solutions of the linear ______________________ equation

  is then $y_1 + y_2$ also a solution? And how about $5y_1$?

- Consider the following exercise:
  (1) Guess a simple solution of $y'' + y = 7$.

  (2) Find fundamental solutions of $y'' + y = 0$.

  (3) Now give 3 different solutions of $y'' + y = 7$.

2.3.1. The General Solution Theorem.

Theorem 1. (General Solution (NH)) If y_1 and y_2 are fundamental solutions of

where ____, and __________ is one solution of __________________, then all solutions of the ______________________________ equation ________________ are

Remark: The ***general solution*** of $L(y) = f$ is

where y_p solves $L(y_p) = f$ and y_1, y_2 are fundamental solutions of $L(y) = 0$.

Proof of Theorem 1:

□

2.3.2. **The Undetermined Coefficients Method.**

Example 1 (Guessing Solutions): If a_1, a_0 are arbitrary constants, guess a function y_p solution of

$$y'' + a_1\, y' + a_0\, y = 3\, e^{2t}$$

Solution:

$\lhd$

Summary of the Undetermined Coefficients Method:

Problem Find __________ solution of $L(y_p) =$ ____________,
where $L(y) = y'' + a_1 y' + a_0 y$.

(1) **First Guess:** Given a simple ____________, guess _______________.

$f(t)$ (Source) (K, m, a, b, given.)	$y_p(t)$ (Guess) (k not given.)
Ke^{at}	ke^{at}
Either t^m or $K_m t^m + \cdots + K_0$	$k_m t^m + \cdots + k_0$
$\cos(bt)$ and/or $\sin(bt)$	$k_1 \cos(bt) + k_2 \sin(bt)$

(2) **Possible Second Guess:** If ___________ satisfies _______________________,
then change the guess to _______________________.

(3) **Possible Third Guess:** If ___________ satisfies _______________________,
then change the guess to _________________________.

(4) **Find the Undetermined Coefficients:** From ______________________ get ________,
where y_p is _______________________________________

Example 2. (First Guess Right): Find all solutions to the nonhomogeneous equation

$$y'' - 3y' - 4y = 3\,e^{2t}.$$

Solution:

◁

Example 3. (First Guess Wrong): Find all solutions to the nonhomogeneous equation

$$y'' - 3y' - 4y = 3\,e^{4t}.$$

Solution:

$\triangleleft$

2.4. Nonhomogeneous Equations

Section Objective(s):

Part 1:

- The General Solution Theorem(NH).
- The Undetermined Coefficients Method.

Part 2:

- The Variation or Parameters Method.

Remarks:

- Recall: The general solution of $L(y) = f$ is

  where

- The Undetermined Coefficients Method (UCM) is a way ____________________.
- The Variation of Parameters Method (VPM) gives ____________________________.
- VPM works on _____________________________ equations than the UCM.
- VPM works on ____
- VPM usually ______________________________ to implement than the UCM.

2.4.1. **The Variation of Parameters Method.**

Theorem 1. (Variation of Parameters) A particular solution to the equation

with $L(y) = y'' + a_1(t)\,y' + a_0(t)\,y$ and a_1, a_0, f continuous functions, is given by

where y_1, y_2 are fundamental solutions of $L(y) = 0$ and ________________ are

where _____________ is the _________________________ of y_1 and y_2.

Remarks:

- The ***Wronskian*** of functions y_1 and y_2 is

- If y_1 and y_2 are fundamental solutions of $y'' + a_1(t)\,y' + a_0(t)\,y = 0$,

  then ____

Proof of Theorem 1:

□

Remark: The integration constants in __________ can always be chosen ________.

Example 1: Find the general solution of the nonhomogeneous equation

$$y'' + 4y = -5\csc(2t).$$

Solution:

◁

EXAMPLE 2.: Find a particular solution to the differential equation

$$t^2 y'' - 2y = 3t^2 - 1,$$

knowing that $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2 y'' - 2y = 0$.

SOLUTION:

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